# Relativistic Quantum Mechanics

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# PLANE WAVE SOLUTIONS OF KG

- To get a feeling for what the KG equation describes, let us look for separable solutions

.
$$\Psi(r,t) = \psi(\vec{r})T(t) \quad (1)$$

- Substituting this value in K G equation, we get

$$-\hbar^2 \frac{\ddot{T}}{T} = \frac{m^2c^4\psi - \hbar^2c^2\nabla^2\psi}{\psi} = E^2 \quad (2)$$

where we introduced a separation constant $E^2$ independent of both ~r and t and having dimensions of energy - squared. $\Psi(r)$ must be an eigenfunction of the Laplacian (i.e., satisfy the Helmholtz equation)

$$\left(-\hbar^2c^2\nabla^2 + m^2c^4\right)\psi = E^2\psi. \quad (3)$$

- The operator $-\hbar^2c^2\nabla^2{+}m^2c^4$ is a +ve operator and coincides with the hamiltonian of a non-relativistic particle in a constant potential. So, its eigenfunctions are free particle energy eigenstates (e.g. plane waves propagating in any direction). So, the separation constant $E^2$ must be positive, which justifies the notation $E^2$ with E real. Then, we have

$$T(t) = Ae^{iEt/\hbar} + Be^{-iEt/\hbar} \quad (4)$$

- Now $E^2 - m^2c^4$, being the eigenvalue of a positive operator $-\hbar^2c^2\nabla^2$ must be non-negative. Let us denote the quantity $E^2 - m^2c^4$ by $p^2c^2$, for some positive number $p^2$ . We of course recognize that the above Helmholtz equation arises from the relativistic energy-momentum dispersion relation $E^2 - m^2c^4 = p^2c^2$, upon use of the correspondence rule $p \to -i\hbar\nabla$. The general solution of the Helmholtz equation is a linear combination

$$\psi(\vec{r}) = Fe^{i\vec{p}\cdot\vec{r}/\hbar} + Ge^{-i\vec{p}\cdot\vec{r}/\hbar} \qquad (5)$$

Thus, separable solutions of the KG equation take the form

$$\Psi(\vec{r},t) = \left(Fe^{i\vec{p}\cdot\vec{r}/\hbar} + Ge^{-i\vec{p}\cdot\vec{r}/\hbar}\right)\left(Ae^{iEt/\hbar} + Be^{-iEt/\hbar}\right) \qquad (6)$$

- These solutions are bounded over all space at all times and could potentially describe the amplitude of some disturbance.
- A peculiar feature of above solution is that for a fixed momentum vector $\vec{p}$, there are plane waves that moves in the direction of $\vec{p}$ and in the opposite direction.
- From classical mechanics as well as non-relativistic quantum mechanics, a particle with given momentum moves in the direction of the momentum vector
- This is reflection of the fact that we started with condition $E^2 - m^2c^4 = p^2c^2$, which includes both positive and negative energies for a given momentum vector.
- This problem did not arise for the Schrodinger equation as it is first order in time, while the KG equation is second order in time.

- Another way of looking at this: The KG equation admits (check by substitution) plane wave solutions $e^{-\frac{i}{\hbar}(Et-\vec{p}.\vec{x})}$ where $\vec{k}=\vec{p}/\hbar$ is an arbitrary wave vector and

$$E=\pm\sqrt{m^2c^4+\hbar^2c^2\vec{k}^2} \qquad (7)$$

- We may call the mode with $E>0$ or $\omega=E/\hbar>0$ or a positive energy/frequency mode and one with $\omega<0$ a negative energy mode.

- This nomenclature is somewhat arbitrary since we could have written the plane wave as $e^{\frac{i}{\hbar}(Et+\vec{p}.\vec{x})}$. Independent of convention, for every plane wave with `energy' E, there is one with energy -E. In this sense, the spectrum of energies of the massive KG equation is continuous and comes in two disjoint pieces $(-\infty,-mc^2]\cup[mc^2,\infty)$. So the energy spectrum is not bounded below, there is no least value of E.

- Calculate the group speed $v_g=\frac{d\omega}{dk}$ of disturbances that propagate according to the dispersion relation $E=\hbar\omega(k)=\sqrt{m^2c^4+\ \hbar^2k^2c^2}$ show that the group speed is less than the speed of light and approaches c when $m\ \to 0$. The group speed is the speed at which signals propagate.

- The phase speed $v_p = \frac{\omega}{k}$ can exceed the speed of light. Physical signals do not travel at the phase speed. The different planes waves that combine to form a wave packet can have phase speeds that exceed the speed of light they destructively interfere at most locations except in the vicinity of the wave packet, which travels at the group speed.

- One option is to simply disallow the negative energy solutions. For example, we might implement this for plane waves by allowing only those initial conditions which ensure that the wave moves in the direction of the momentum vector, ensuring that E > 0. Within the context of the KG equation, this is seemingly ok, since the particle will then remain in that stationary state for ever. However, under the influence of external perturbations, the particle could make a transition to a lower energy state.

- Since there is no ground state, the particle could keep dropping down in energy while emitting radiation. The system is unstable to perturbations as it does not have a ground state. This is problematic since we could extract an infinite amount of energy from such a particle as it makes transitions to states of arbitrarily negative energy.

- Despite this difficulty with trying to interpret solutions of the KG equation as the wave function of a particle, the equation exhibits several physically desirable features, such as a local conservation law and Lorentz invariance, which we describe next.

## NON-RELATIVISTIC LIMIT

- To obtain non-relativistic limit of the KG equation, one cannot do this by simply putting $c = \infty$ in the KG equation.

- Classically, a nonrelativistic situation is one where the energy is mostly rest energy. For a free particle i.e. $E = mc^2 + KE \approx mc^2 + \frac{p2}{2m}$

- In this case, the primary time dependence of a plane wave $\psi(\vec{x}, t) = e^{-\frac{i}{\hbar}(Et - \vec{p}.\vec{x})}$ is given by putting $E \approx mc^2$ .

- Of course, there would be some residual time dependence due to the remaining energy. So to facilitate taking the non-relativistic limit, let us change variables to a new wave function $\phi(\vec{x}, t)$

$$\psi(r,t) = e^{-imc^2t/\hbar}\phi(r,t)$$

- We have in mind that the factor $e^{-imc^2t/\hbar}$ takes care of the fast time dependence (high frequency) and $\phi(\vec{x}, t)$only has a residual slow time dependence. Putting this form in KG i. e. $E^2\psi(\vec{x}, t) = (m^2c^4 + p^2c^2)\psi(\vec{x}, t)$, one finds that satisfies $\phi(\vec{x}, t)$

$$i\hbar\dot{\phi} - \frac{\hbar^2}{2mc^2}\ddot{\phi} = -\frac{\hbar^2}{2m}\nabla^2\phi$$

- So far, we have made no approximation.

- Now we may take a non-relativistic limit by letting $c = \infty$ dependence and we get the usual free particle SE.

- An energy eigenstate is then of the form $\psi(\vec{x}, t) = e^{-\frac{i}{\hbar}(E_{nr}t - \vec{p}.\vec{x})}$, where $E_{nr} = \hbar^2 k^2/2m$.

- Thus, for an energy eigenstate, the original wave function is $\psi(\vec{x}, t) = e^{-\frac{i}{\hbar}(Et - \vec{p}.\vec{x})}$ where $E = mc^2 + E_{nr}$

# COUPLING TO ELECTROMAGNETIC FIELD OR KG EQUATION IN ELECTROMAGNETIC FIELD

- We can study the KG equation in the presence of an electromagnetic field defined by the scalar and vector potentials $\varphi, \vec{A}$ in the same way as we did for the Schrodinger equation.

- We apply the `minimal coupling' prescription $E \to E - e\,\varphi$ $\vec{P} \to \vec{P} - e\,\vec{A}$ (Where E is energy and $\vec{P}$ is momentum) to the relativistic energy momentum dispersion relation $E^2 = m^2c^4 + p^2c^2$

- $A_{\mu} = (\frac{\varphi}{c}, -\vec{A})$ transform under Lorentz transformations in the same manner as $A_{\mu} = (\frac{E}{c}, -\vec{p})$ i.e. as the covariant components of a 4-vector.

- To get a wave equation we then use the correspondence rule $E \to i\hbar\frac{\partial}{\partial t}$, $p \to -i\hbar\nabla$ and treat $\varphi, \vec{A}$ as multiplication operators on the wave function $\psi(\vec{x}, t)$.

$$\left(i\hbar\frac{\partial}{\partial t} - e\phi\right)^2\psi = c^2\left(-i\hbar\nabla - e\vec{A}\right)^2\psi + m^2c^4\psi$$

- This equation can be written manifestly in Lorentz invariant form. Recall that KG could be written as $(p^{\mu}p_{\mu} - m^2c^2)\psi(\vec{x},t) = 0$. Coupling to an electromagnetic field simply means we replace $p_{\mu} \rightarrow p_{\mu} - eA_{\mu}$

- Check that this is the same as the above equation.

- We will say more about the electromagnetic interaction of a relativistic particle when we discuss the Dirac equation.

References:

*1) A Text Book of Quantum Mechanics by P.M. Mathews and K. Venkatesan*

*2) Quantum Mechanics by L.I. Schiff*

*3) Lectures on relativistic quantum mechanics by Govind S. Krishnaswami, Chennai Mathematical Institute*