(4) Time Reversal Property :

If $x(n) \xleftrightarrow{Z} X(z)$

then $x(-n) \xleftrightarrow{Z} X(z^{-1})$.

Proof :- Ac to definition of Z transform

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\therefore Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

Put $l = -n$, the limits will change

when $n = -\infty$, $l = \infty$ and when $n = \infty$, $l =$

$$\therefore Z\{x(-n)\} = \sum_{l=-\infty}^{\infty} x(l) z^{l}$$

$$\therefore Z\{x(-n)\} = \sum_{l=-\infty}^{\infty} x(l)(z^{-1})^{-l}$$

Comparing this eq. with defenitation of Z-transform

$$Z\{x(-n)\} = X(z^{-1}).$$

3.3.4.1 Solved Examples

Ex. 3.3.21 : Obtain the Z-transform of signal $x(n) = u(-n)$.

Soln. :

We have the Z-transform of unit step,

$$Z\{u(n)\} = \frac{Z}{Z-1} \qquad ROC : |Z| > 1$$

According to the time reversal property,

$$Z\{x(-n)\} = X(Z^{-1})$$

Thus Z-transform of $x(-n)$ is obtained by replacing Z by Z^{-1} in Equation (1)

$$\therefore \quad Z\{u(-n)\} = \frac{Z^{-1}}{Z^{-1}-1} \qquad ROC : |Z^{-1}| > 1$$

Multiplying numerator and denominator by Z we get,

$$Z\{u(-n)\} = \frac{1}{1-Z} \qquad ROC : |Z^{-1}| > 1$$

Here ROC is $|Z^{-1}| > 1$

$$\therefore \quad \left|\frac{1}{Z}\right| > 1 \text{ that means } |Z| < 1.$$

$$\therefore \quad u(-n) \xleftrightarrow{Z} \frac{1}{1-z} \quad \text{ROC} : |z| < 1$$

Thus ROC is interior part of circle having radius $r = 1$. The circle of radius = 1 is also called as unit circle. Thus ROC is interior part of unit circle as shown in Fig. P. 3.3.21.

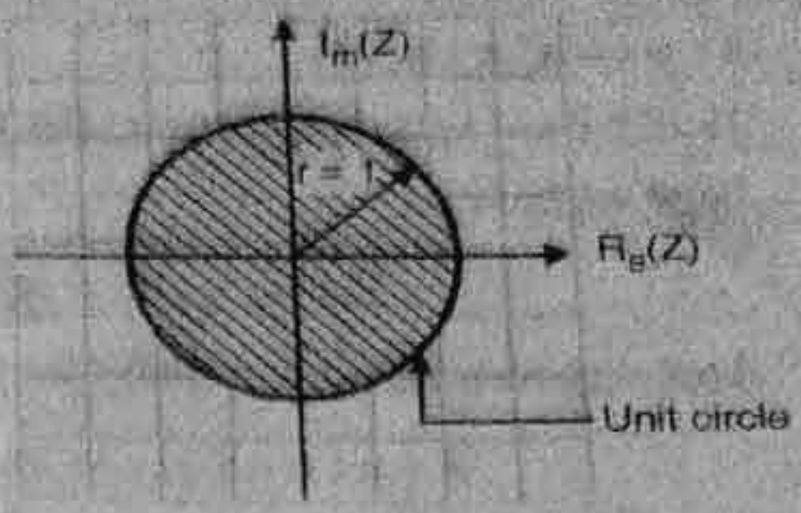

Fig. P. 3.3.21 : ROC of u (– n)

Ex. 3.3.22 : Determine Z-transform and ROC of :

$$x(n) = \left(\frac{1}{2}\right)^n u(-n)$$

Soln. :

We have $Z[u(-n)] = \frac{1}{1-Z}$

ROC : $|Z| < 1$

Here the given signal is $\left(\frac{1}{2}\right)^n u(-n)$. According to scaling property we have,

$$Z[a^n x(n)] = X\left(\frac{Z}{a}\right).$$

That means we will have to replace Z by $\frac{Z}{a}$. In this example $a = \frac{1}{2}$.

Thus, $Z\left\{\left(\frac{1}{2}\right)^n u(-n)\right\} = \frac{1}{1-\frac{Z}{1/2}}$

ROC : $\left|\frac{Z}{1/2}\right| < 1$

$$\therefore \quad Z\left\{\left(\frac{1}{2}\right)^n u(-n)\right\} = \frac{1}{1-2Z} \quad \text{ROC} : |Z| < \frac{1}{2}$$

Thus ROC is interior part of circle having radius r = 1/2. This ROC is similar to Fig. P. 3.3.14. Only difference is here, r = 1/2.

Ex. 3.3.23 : Find Z-transform of following signals and comment on the results.

(i) $x(n) = a^n u(n-1)$ (ii) $x(n) = a^n u(-n-1)$

(iii) $x(n) = -(a^n) u(n-1)$ (iv) $x(n) = -(a^n) u(-n-1)$

Solu :

(i) $x(n) = a^n u(n-1)$

Here $u(n-1)$ is shifted unit step. First we will obtain $Z\{u(n-1)\}$.

Now we have,

$$Z\{u(n)\} = \frac{Z}{Z-1} \qquad \text{ROC} : |Z| > 1$$

Applying time shifting property we can write,

$$Z\{u(n-1)\} = Z^{-1} \times \frac{Z}{Z-1} = \frac{1}{Z-1} \qquad \text{ROC} : |Z| > 1$$

Now we have to obtain $Z\{a^n u(n-1)\}$.

Applying scaling property we get,

$$Z\{a^n u(n-1)\} = \frac{1}{\frac{Z}{a}-1} \qquad \text{ROC} : \left|\frac{Z}{a}\right| > 1$$

$$\boxed{Z\{a^n u(n-1)\} = \frac{a}{Z-a} \qquad \text{ROC} : |Z| > a}$$

This ROC is exterior part of circle having radius, $r = a$.

(ii) $x(n) = a^n u(-n-1)$

First we will obtain $Z\{u(-n-1)\}$. Here $u(-n)$ is folded unit step. The Z-transform of folded unit step is given by,

$$Z\{u(-n)\} = \frac{1}{1-Z} \qquad \text{ROC} : |Z| < 1$$

Now recall time shifting property, it is

$$\text{If} \quad x(n) \xleftrightarrow{Z} X(Z) \text{ then } x(n-k) \xleftrightarrow{Z} Z^{-k} X(Z).$$

Here we have folded sequence. For folded sequence this property can be written as,

$$x(-n-k) \xleftrightarrow{Z} Z^{+k} X(Z)$$

This is because folded sequence is mirror image of original sequence.

Thus we can write,

$$Z\{u(-n-1)\} = Z^{+1} \cdot Z\{u(-n)\} = Z^{1} \cdot \frac{1}{1-Z} \qquad \text{ROC}:$$

$$\boxed{Z\{u(-n-1)\} = \frac{Z}{1-Z} \qquad \text{ROC} : |Z| < 1}$$

Now we have to obtain $Z\{a^n u(-n-1)\}$. Using scaling property we get,

$$Z\{a^n u(-n-1)\} = \frac{Z/a}{1-Z/a} \qquad \text{ROC} : \left|\frac{Z}{a}\right| < 1$$

$$\boxed{Z\{a^n u(-n-1)\} = \frac{Z}{a-Z} \qquad \text{ROC} : |Z| < |a|}$$