- **Relationship between** $E(L_S), E(L_q), E(W_S), E(W_q)$

i) $E(L_S) = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$ and $E(W_S) = \frac{1}{\mu(1-\rho)} = \frac{1}{\mu-\lambda}$ therefore $E(L_S) = \lambda E(W_S)$

ii) $E(L_q) = \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)}$ and $E(W_q) = \frac{\rho}{\mu(1-\rho)} = \frac{\lambda}{\mu(\mu-\lambda)}$ **,**Therefore, $E(L_q) = \lambda E(W_q)$

iii) $E(W_S) = \frac{1}{\mu(1-\rho)} = \frac{1}{\mu-\lambda}$ and $E(W_q) = \frac{\rho}{\mu(1-\rho)} = \frac{\lambda}{\mu(\mu-\lambda)}$, Therefore $E(W_S) = E(W_q) + \frac{1}{\mu}$

Model -2 (General Erlang Queuing model)

This model is similar to model 1 but difference is only that mean arrival rate and service rate are depend on n not constant.

- λ_n = mean effective arrival rate (number of customers arriving per unit time).
- μ_n = mean service rate per busy server (number of customers served per unit tine)
- **Step 1: to find the system of steady state equations.**

Proceeding similarity as in Model 1, the Probability of n units in the system at time $(t+\Delta t)$ is given by

(i) (n-1) units in the system at time t with probability $P_{n-1}(t)$

Probability of one arrival in time Δt i.e. $P_1(\Delta t) = \lambda_{n-1}\Delta t$

Probability of no service in time Δt i.e. $\varphi_{\Delta t}(0) = 1-\mu_{n-1}\Delta t$

The probability in this case is $P_{n-1}(t)\lambda_{n-1}.\Delta t.(1-\mu_{n-1}.\Delta t)$

(ii) n units in the system at time t with probability $P_n(t)$

Probability of no arrival in time Δt i.e. $P_0(\Delta t) = 1 - \lambda_n \Delta t$

Probability of no service in time Δt i.e. $\varphi_{\Delta t}(0) = 1 - \mu_n \Delta t$

The probability in this case is $P_n(t)(1-\lambda_n.\Delta t).(1-\mu_n.\Delta t)$

(iii) (n+1) units in the system at time t with probability $P_{n+1}(t)$

Probability of no arrival in time Δt i.e. $P_0(\Delta t) = 1 - \lambda_{n+1} \Delta t$

Probability of one service in time Δt i.e. $\varphi_{\Delta t}(0) = \mu_{n+1} \Delta t$

The probability in this case is $P_{n+1}(t)(1-\lambda_{n+1}.\Delta t).\mu_{n+1}.\Delta t$

Since all the above three cases are mutually exclusive, therefore $P_n(t+\Delta t)$, the probability of n units in the system at time $(t+\Delta t)$ is obtained by adding the probabilities in the above three cases.

$$P_n(t+\Delta t) = P_{n-1}(t)\lambda_{n-1}.\Delta t.(1-\mu_{n-1}.\Delta t) + P_n(t)(1-\lambda_n.\Delta t).(1-\mu_n.\Delta t) + P_{n+1}(t)(1-\lambda_{n+1}.\Delta t).\mu_{n+1}.\Delta t$$

$$\frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \lambda_{n-1}P_{n-1}(t) - (\lambda_n + \mu_n)P_n(t) + \mu_{n+1}P_{n+1}(t) + \frac{\text{higher order of } \Delta t}{\Delta t} \qquad (1)$$

Similarly there is no units in the system at time $(t+\Delta t)$ in the following two ways.

(i) No units in the system at time t with probability $P_0(t)$

Probability of no arrival in time Δt i.e. $P_0(\Delta t) = 1 - \lambda_0 \Delta t$

The probability in this case is $P_0(t)(1-\lambda_0.\Delta t)$

(ii) one units in the system at time t with probability $P_1(t)$

Probability of no arrival in time Δt i.e. $P_0(\Delta t) = 1 - \lambda_1 \Delta t$

Probability of one service in time Δt i.e. $\varphi_{\Delta t}(0) = \mu_1 \Delta t$

The probability in this case is $P_1(t)(1-\lambda_1.\Delta t).\mu_1.\Delta t$

Adding the probabilities in the above two cases.

$$P_0(t+\Delta t) = P_0(t)(1-\lambda_0.\Delta t) + P_1(t)(1-\lambda_1.\Delta t).\mu_1.\Delta t$$

$$\frac{P_0(t+\Delta t)-P_0(t)}{\Delta t} = -\lambda_0 P_0(t) + \mu_1 P_1(t) + \frac{higher\ order\ of\ \Delta t}{\Delta t} \qquad (2)$$

Taking limit $\Delta t \to 0$, from (1) & (2)

$$P_n^{'}(t) = \lambda_{n-1} P_{n-1}(t) - (\lambda_n + \mu_n) P_n(t) + \mu_{n+1} P_{n+1}(t) \qquad (3)$$

$$P_0^{'}(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t) \qquad (4)$$

When the system reaches steady state i.e.

$$\lim_{t\to\infty} P_n(t) = P_n \ and \lim_{t\to\infty} P_n^{'}(t) = 0$$

Therefore $\lambda_{n-1} P_{n-1}(t) - (\lambda_n + \mu_n) P_n(t) + \mu_{n+1} P_{n+1}(t) = 0$ (5)

$$-\lambda_0 P_0(t) + \mu_1 P_1(t) = 0 \qquad (6)$$

These equations (5) and (6) are the steady state equations of the system.

To solve the steady state equations (5) and (6) obtained in step 1.

From (6) $P_1 = \dfrac{\lambda_0}{\mu_1} P_0$

From(5) $P_{n+1}(t) = -\dfrac{\lambda_{n-1}}{\mu_{n+1}} P_{n-1}(t) + \dfrac{(\lambda_n + \mu_n)}{\mu_{n+1}} P_n(t) = -\dfrac{\lambda_{n-1}}{\mu_{n+1}} P_{n-1}(t) + \left(\dfrac{\lambda_n}{\mu_{n+1}} + \dfrac{\mu_n}{\mu_{n+1}}\right) P_n(t)$

put n=1 $P_2(t) = -\dfrac{\lambda_0}{\mu_2} P_0(t) + \dfrac{(\lambda_1 + \mu_1)}{\mu_2} P_1(t) = -\dfrac{\lambda_0}{\mu_2} P_0(t) + \left(\dfrac{\lambda_1}{\mu_2} + \dfrac{\mu_1}{\mu_2}\right) P_1(t)$

$$P_2(t) = -\frac{\lambda_0}{\mu_2}\frac{\mu_1}{\lambda_0}P_1 + \frac{(\lambda_1+\mu_1)}{\mu_2}P_1(t) = -\frac{\mu_1}{\mu_2}P_1(t) + \left(\frac{\lambda_1}{\mu_2}+\frac{\mu_1}{\mu_2}\right)P_1(t) = \frac{\lambda_1}{\mu_2}P_1(t) = \frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}P_0$$

Similarly, $P_3(t) = \frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\frac{\lambda_2}{\mu_3}P_0$

Proceeding in this way, we have

$$P_n(t) = \frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\frac{\lambda_2}{\mu_3}..........\frac{\lambda_{n-1}}{\mu_n}P_0 \qquad (7)$$

We know that $\sum_{n=1}^{\infty} P_N = 1 \Rightarrow P_0 + P_1 + P_2 + P_3............ = 1$

$$1+\frac{\lambda_0}{\mu_1}P_0+\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}P_0+\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\frac{\lambda_2}{\mu_3}P_0.................. = 1$$

$$P_0 = \frac{1}{S} \qquad (8)$$

$$S = 1+\frac{\lambda_0}{\mu_1}P_0+\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}P_0+\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\frac{\lambda_2}{\mu_3}P_0................ \qquad (9)$$

If S is divergent then P_0=0, which is meaningless.

Therefore, we assume that S is convergent series.

Particular cases:

Case 1: $\left.\begin{matrix}\lambda_n = \lambda \\ \mu_n = \mu\end{matrix}\right\}$ (both independent of n)

$$S = 1+\frac{\lambda}{\mu}P_0+\left(\frac{\lambda}{\mu}\right)^2 P_0+\left(\frac{\lambda}{\mu}\right)^3 P_0................$$

$$= \frac{1}{1-\frac{\lambda}{\mu}} = \frac{1}{1-\rho}$$

$$P_0 = 1-\rho$$
$$P_n = \rho^n(1-\rho)$$

These correspond to the result of Model 1.

Case 2: $\left.\begin{array}{l}\lambda_n = \dfrac{\lambda}{n+1} \\ \mu_n = \mu\end{array}\right\}$

In this case λ_n decreases as n increases, i.e. arrival rate decreases with the increase in queue length and rate of service is independent of n.

This is called the case of "Queue with discouragement".

Substituting the values of λ_n and μ_n in (9)

$$S = 1 + \frac{\lambda}{\mu} P_0 + \frac{\lambda^2}{1.2\mu^2} P_0 + \frac{\lambda^3}{2.3\mu^3} P_0 \ldots\ldots\ldots\ldots\ldots\ldots$$
$$= 1 + \rho + \frac{\rho^2}{2!} + \frac{\rho^3}{3!} \ldots\ldots = e^{\rho}$$

From (8)

$$P_0 = \frac{1}{S} = e^{-\rho}$$

$$P_1 = \frac{\lambda_0}{\mu_1} P_0 = \rho e^{-\rho}$$

$$P_2(t) = \frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2} P_0 = \frac{\rho^2}{2!} e^{-\rho}$$

$$P_2(t) = \frac{\rho^n}{n!} e^{-\rho} \qquad (11)$$

Which implies that $P_n(t)$ follows the poisson distribution.

Case 3: If $\lambda_n = n$ and $\mu_n = n\mu$, This is known as a problem of "queue with infinite number of channels"

Substituting the values of λ_n and μ_n in (9)

$$S = 1 + \frac{\lambda}{\mu} P_0 + \frac{\lambda^2}{1.2\mu^2} P_0 + \frac{\lambda^3}{2.3\mu^3} P_0 \ldots\ldots\ldots\ldots\ldots$$

$$= 1 + \rho + \frac{\rho^2}{2!} + \frac{\rho^3}{3!} \ldots\ldots = e^{\rho}$$

$$P_0 = \frac{1}{S} = e^{-\rho} \qquad (12)$$

$$P_1 = \frac{\lambda_0}{\mu_1} P_0 = \rho e^{-\rho}$$

$$P_2(t) = \frac{\lambda_0}{\mu_1} \frac{\lambda_1}{\mu_2} P_0 = \frac{\rho^2}{2!} e^{-\rho}$$

$$P_n(t) = \frac{\rho^n}{n!} e^{-\rho} \qquad (13)$$

Which implies that $P_n(t)$ follows the poisson distribution in this case also.

(a) To find Expected (or average) number of customer in the system (customers in the line plus the customer being served)

$$E(L_S) = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n \frac{\rho^n}{n!} e^{-\rho} = \rho \sum_{n=0}^{\infty} \frac{\rho^{n-1}}{(n-1)!} e^{-\rho} = \rho e^{-\rho} e^{\rho} = \rho$$

$$E(L_S) = \rho = \frac{\lambda}{\mu}$$

Now with the help of relations (A), (B),and (C), $E(L_q)$ $E(W_q)$ and $E(W_S)$ may be obltained.