Multiple Regression Analysis

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Linear Regression Analysis:-

Let $Y = \alpha + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_m X_m$ be the relationship b/w Y and X. Let us take the obs^n's on Y on X as

$$Y_1 = x_{11} \quad x_{21} \quad x_{31} \quad \cdots \quad x_{m1}$$
$$Y_2 = x_{12} \quad x_{22} \quad x_{32} \quad \cdots \quad x_{m2}$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$Y_n = x_{1n} \quad x_{2n} \quad x_{3n} \quad \cdots \quad x_{mn}$$

Now, we expect that

$$Y_i = \alpha + \sum_{j=1}^{m} \beta_j x_{ji} \quad ; \; i = 1, 2, \cdots n$$

$$\therefore \quad E(Y \mid x_1, x_2, \cdots x_m) = \alpha + \sum_{j=1}^{m} \beta_j x_{ji}$$

This is the linear model.

but it does not happen in practice and we encounter chance factor resulting in some error or residual sum of square. i.e.

$$Y_i = \alpha + \sum_{j=1}^{m} \beta_j x_{ji} + e_i$$

and we define

$$S^2 = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} \left(Y_i - \alpha - \sum_{j=1}^{m} \beta_j x_{ji} \right)^2$$

error sum of square. And our purpose is to minimize S^2 w.r. to the par α and β_j's and finds estimate of α and β_j's in the terms of obs^n's $y_1, y_2, \cdots y_n$ and x. This is also known as least square theory of Linear Reg.

Multiple Regression Analysis :- Let $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}$, consider the linear reg. of x_1 on $x_2, x_3, \dots x_p$. Then the model is

$$x_{1.23\dots p} = E[x_1 | x_2 \dots x_p]$$
$$= \alpha + \beta_2 x_2 + \dots + \beta_p x_p$$

Note that, x_1 is separated var. generally depending on $x_2, x_3, \dots x_p$ where as $x_{1.23\dots p}$ is the conditional expectation of $x_1 | x_2 x_3 \dots x_p$. Now, in practice since x_1 is not simpally $x_{1.23\dots p}$, therefore we have

$$x_1 = x_{1.23\dots p} + e_{1.23\dots p}$$

Thm :- If the var-cov matrix of variables x_i : $i = 1, 2, \dots p$ exist and

$$\begin{vmatrix} \sigma_{22} & \sigma_{23} & \dots & \sigma_{2p} \\ \sigma_{32} & \sigma_{33} & \dots & \sigma_{3p} \\ \vdots & & & \vdots \\ \sigma_{p2} & \sigma_{p3} & \dots & \sigma_{pp} \end{vmatrix} = 0$$

Further if the reg. of x_1 on $x_2, x_3 \dots x_p$ is linear then the constant α and β_j's : $j = 2, 3, \dots p$ occuring in the linear reg. eq^n x_1 on $x_2, x_3 \dots x_p$ are given by $\alpha = \frac{\sigma_1}{R_{11}} \sum_{j=1}^{p} \frac{R_{1j}}{\sigma_{jj}} \mu_j$ and $\beta_j = -\frac{\sigma_1}{\sigma_j} \frac{R_{1j}}{R_{11}}$ $\forall j = 2, \dots p$

where, $\sigma_j = \sqrt{\sigma_{jj}}$ and R_{ij} is the cofactor of δ_{ij} in the determinent of mat. of corr.

Proof :- First we note that the means μ_j's and x_j's necessarily exist, since their var. exist. Now, for simplicity but without loss of generality we assume that the joint dist^n of $x_1, x_2, \dots x_p$ be absolutely cts. Then, the eq^n

$$E[x_1 | x_2, x_3 \dots x_p] = \alpha + \beta_2 x_2 + \beta_3 x_3 + \dots + \beta_p x_p$$

$$\Rightarrow \int_R x_1 f(x_1 | x_2 x_3 \dots x_p) \, dx_1 = \alpha + \beta_2 x_2 + \beta_3 x_3 + \dots + \beta_p x_p \quad \text{---(1)}$$

Now, on multiplying both sides of eq^n (1) by the joint dist^n of $x_2, x_3 \dots x_p$ say $g(x_2, x_3 \dots x_p)$ and integrating w.r. to $x_2, x_3 \dots x_p$, we get

$$\Rightarrow \int_R x_1 f(x_1 | x_2 x_3 \dots x_p) \times g(x_2 x_3 \dots x_p) \, dx_1 dx_2 \dots dx_p$$
$$= \alpha + \beta_2 \int_{x_2} \int_{x_3} \dots \int_{x_p} x_2 \, g(x_2 x_3 \dots x_p) \, dx_2 dx_3 \dots dx_p$$
$$+ \dots + \beta_p \int_{x_2} \int_{x_3} \dots \int_{x_p} x_p \, g(x_2 x_3 \dots x_p) \, dx_2 dx_3 \dots dx_p$$

$$\Rightarrow E(x_1) = \alpha + \beta_2 E(x_2) + \beta_3 E(x_3) + \dots + \beta_p E(x_p)$$

$$\Rightarrow \mu_1 = \alpha + \beta_2 \mu_2 + \beta_3 \mu_3 + \dots + \beta_p \mu_p \quad \text{---(2)}$$

Now on again multiplying both sides of eq^n (1) by $x_j * g(x_2 x_3 \dots x_p)$; $j = 2, 3, \dots p$ successively and integrating w.r. to $x_2 x_3 \dots x_p$, we get

$$E(x_1 x_2) = \alpha E(x_2) + \beta_2 E(x_2^2) + \dots + \beta_p E(x_p x_2)$$
$$E(x_1 x_3) = \alpha E(x_3) + \beta_2 E(x_2 x_3) + \dots + \beta_p E(x_p x_3)$$
$$\vdots \quad \text{---(3)}$$

$$E(X_1X_p) = \alpha E(X_p) + \beta_2 E(X_2X_p) + \cdots + \beta_p E(X_p^2)$$

Further, we multipling on both sides of eqn (2) by μ_j ; $j = 2, 3 \cdots p$ successively we get

$$\left.\begin{aligned}
\mu_1\mu_2 &= \alpha\mu_2 + \beta_2\mu_2^2 + \cdots + \beta_p\mu_p\mu_2 \\
\mu_1\mu_3 &= \alpha\mu_3 + \beta_2\mu_2\mu_3 + \cdots + \beta_p\mu_p\mu_3 \\
&\ \vdots \\
\mu_1\mu_p &= \alpha\mu_p + \beta_2\mu_2\mu_p + \cdots + \beta_p\mu_p^2
\end{aligned}\right\} \quad -(4)$$

Then on subtracting eqn (4) from the corresponding eqn (3), we get

$$E(X_1X_2) - \mu_1\mu_2 = \beta_2[E(X_2^2) - \mu_2^2] + \beta_3[E(X_2X_3) - \mu_2\mu_3] + \cdots + \beta_p[E(X_pX_2) - \mu_p\mu_2]$$

$$E(X_1X_3) - \mu_1\mu_3 = \beta_2[E(X_2X_3) - \mu_2\mu_3] + \cdots + \beta_p[E(X_pX_3) - \mu_p\mu_3]$$

$\vdots$

$$E(X_1X_p) - \mu_1\mu_p = \beta_2[E(X_2X_p) - \mu_2\mu_p] + \cdots + \beta_p[E(X_p^2) - \mu_p^2]$$

which provides us,

$$\sigma_{12} = \beta_2\sigma_{22} + \beta_3\sigma_{23} + \cdots + \beta_p\sigma_{2p}$$

$$\sigma_{13} = \beta_2\sigma_{32} + \beta_3\sigma_{33} + \cdots + \beta_p\sigma_{3p}$$

$\vdots$

$$\sigma_{1p} = \beta_2\sigma_{p2} + \beta_3\sigma_{p3} + \cdots + \beta_p\sigma_{pp}$$

which can be written as in the matrix form

$$\begin{bmatrix}\sigma_{12}\\ \sigma_{13}\\ \vdots \\ \sigma_{1p}\end{bmatrix} = \begin{bmatrix}\sigma_{22} & \sigma_{23} & \cdots & \sigma_{2p}\\ \sigma_{32} & \sigma_{33} & \cdots & \sigma_{3p}\\ \vdots & & & \\ \sigma_{p2} & \sigma_{p3} & \cdots & \sigma_{pp}\end{bmatrix}\begin{bmatrix}\beta_2\\ \beta_3\\ \vdots \\ \beta_p\end{bmatrix} \quad — (5)$$

Now, we solve eqn (5) for β_j's, $j = 2, 3, \cdots p$ by using cramer-rule as follows

$$\beta_j = \frac{\begin{vmatrix}\sigma_{22} & \sigma_{23} & \cdots & \sigma_{2(j-1)} & \sigma_{12} & \sigma_{2(j+1)} & \cdots & \sigma_{2p}\\ \sigma_{32} & \sigma_{33} & \cdots & \sigma_{3(j-1)} & \sigma_{13} & \sigma_{3(j+1)} & \cdots & \sigma_{3p}\\ \vdots & & & & & & & \\ \sigma_{p2} & \sigma_{p3} & \cdots & \sigma_{p(j-1)} & \sigma_{1p} & \sigma_{p(j+1)} & \cdots & \sigma_{pp}\end{vmatrix}}{\begin{vmatrix}\sigma_{22} & \sigma_{23} & \cdots & \sigma_{2j} & \cdots & \sigma_{2p}\\ \sigma_{32} & \sigma_{32} & \cdots & \sigma_{3j} & \cdots & \sigma_{3p}\\ \vdots & & & & & \\ \sigma_{p2} & \sigma_{p3} & \cdots & \sigma_{pj} & \cdots & \sigma_{pp}\end{vmatrix}}$$

$$\Rightarrow \quad \beta_j = -\frac{\Sigma_{1j}}{\Sigma_{11}} \quad \forall\ j = 2, 3, \cdots p \quad —(6)$$

where, Σ_{ij} be the cofactor of σ_{ij} in $|\Sigma|$ and

$$|\Sigma| = \sigma_{11}\sigma_{22}\cdots\sigma_{pp}|R| \quad —(7)$$

Here, R is corr. matrix then R_{ij} will be the cofactor of ρ_{ij} in the det. of corr. matrix.

Now, we have

$$\Sigma_{ij} = \sigma_{11}\sigma_{22}\cdots\sigma_{pp}\frac{R_{ij}}{\sqrt{\sigma_{ii}}\sqrt{\sigma_{jj}}} \quad \forall\ j = 2, 3, \cdots p$$

Thus,

$$\Sigma_{1j} = \sigma_{11}\sigma_{22}\cdots\sigma_{pp}\frac{R_{1j}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{jj}}} \quad \forall\ j = 2, 3, \cdots p$$

Also $\Sigma_{11} = \dfrac{\sigma_{11}\sigma_{22}\cdots\sigma_{pp}R_{11}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{11}}}$

$\Sigma_{11} = \sigma_{22}\sigma_{33}\cdots\sigma_{pp}R_{11}$

Thus, we have from eq^n (6)

$$\beta_j = -\frac{\sigma_{11}\cancel{\sigma_{22}}\cdots\cancel{\sigma_{pp}}R_{1j}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{jj}}}\cdot\frac{1}{\cancel{\sigma_{22}\sigma_{33}\cdots\sigma_{pp}}R_{11}}$$

$$= -\frac{\sigma_{11}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{jj}}}\cdot\frac{R_{1j}}{R_{11}}$$

$$= -\sqrt{\frac{\sigma_{11}}{\sigma_{jj}}}\cdot\frac{R_{1j}}{R_{11}} = \boxed{-\frac{\sigma_1}{\sigma_j}\frac{R_{1j}}{R_{11}} = \beta_j} \quad (8)$$

Proved.

Thus, from eq^n (2) and (8), we get

$$\mu_1 = \alpha + \beta_2\mu_2 + \beta_3\mu_3 + \cdots + \beta_p\mu_p$$

$$\alpha = \mu_1 - \beta_2\mu_2 - \beta_3\mu_3 - \cdots - \beta_p\mu_p$$

$$= \mu_1 + \frac{\sigma_1}{\sigma_2}\frac{R_{12}}{R_{11}}\mu_2 + \frac{\sigma_1}{\sigma_3}\frac{R_{13}}{R_{11}}\mu_3 + \cdots + \frac{\sigma_1}{\sigma_p}\frac{R_{1p}}{R_{11}}\mu_p$$

$$\boxed{\alpha = \sum_{j=1}^{p}\frac{\sigma_1}{\sigma_j}\frac{R_{1j}}{R_{11}}\mu_j} \quad (9)$$

Proved.

** Remark :- $X_{1.23\cdots p}$ be the estimated value of X_1 then the residual 'e' is given by

$$e = (X_1 - X_{1.23\cdots p}) = (X_1 - \alpha - \beta_2 X_2 - \beta_3 X_3 - \cdots - \beta_p X_p)$$

$$= X_1 - \frac{\sigma_1}{R_{11}}\sum_{j=1}^{p}\frac{R_{1j}}{\sigma_j}\mu_j + \sum_{j=2}^{p}\frac{\sigma_1}{\sigma_j}\frac{R_{1j}}{R_{11}}X_j$$

$$= \sum_{j=1}^{p}\frac{\sigma_1}{\sigma_j}\frac{R_{1j}}{R_{11}}X_j - \sum_{j=1}^{p}\frac{\sigma_1}{\sigma_j}\frac{R_{1j}}{R_{11}}\mu_j$$

$$= \sum_{j=1}^{p}\frac{\sigma_1}{\sigma_j}\frac{R_{1j}}{R_{11}}(X_j - \mu_j) \qquad \{\because \cancel{\sigma\ \Sigma_{11}R_{11}}\}$$

$$\boxed{e = \sum_{j=1}^{p}\frac{\Sigma_{1j}}{\Sigma_{11}}(X_j - \mu_j)}$$

Some Properties of Residual :-

(i) $E(e) = 0$

Proof :- $E(e) = E\left[\sum_{j=1}^{p}\frac{\Sigma_{1j}}{\Sigma_{11}}(X_j - \mu_j)\right]$

$$= \sum_{j=1}^{p}\frac{\Sigma_{1j}}{\Sigma_{11}}E(X_j - \mu_j)$$

$$= \sum_{j=1}^{p}\frac{\Sigma_{1j}}{\Sigma_{11}}(\mu_j - \mu_j) = 0$$

(ii) $E(X_j \cdot e) = 0$; $j = 2, 3, \cdots p$

Proof :- $e^2 = E(X_1 - X_{1.23\cdots p})^2$

$$= E\left[X_1 - \left(\alpha + \sum_{j=2}^{p}\beta_j X_j\right)\right]^2$$

Then to minimize e^2, we have

$$\frac{\delta e^2}{\delta \beta_j} = 0$$

$$\Rightarrow -2E\left[\left\{X_1 - \left(\alpha + \sum_{j=2}^{p}\beta_j X_j\right)\right\}X_j\right] = 0$$

$$= -2E(eX_j) = 0$$

$$\Rightarrow E(eX_j) = 0$$

(iii) $Var(e) = E(e^2) = E(ee)$

$$= E\left[e\left\{X_1 - \left(\alpha + \sum_{j=2}^{p} \beta_j X_j\right)\right\}\right]$$

$$= E[eX_1] - \alpha E(e) - \sum_{j=2}^{p} \beta_j E(eX_j)$$

$$= E(eX_1) - 0 - 0$$

$$= E[eX_1] - E(e)E(X_1)$$

$$= Cov(X_1, e) \qquad \{\because E(e) = 0\}$$

$\therefore \quad Var(e) = Cov(X_1, e) = E[\{X_1 - \mu_1\} e]$

Now, $E[\{(X_1-\mu_1)e\}] = E\left[(X_1-\mu_1)\sum_{j=1}^{p} \frac{\Sigma_{1j}}{\Sigma_{11}}(X_j - \mu_j)\right]$

$$= \sum_{j=1}^{p} \frac{\Sigma_{1j}}{\Sigma_{11}} E[(X_1-\mu_1)(X_j-\mu_j)]$$

$$= \sum_{j=1}^{p} \frac{\Sigma_{1j}}{\Sigma_{11}} \sigma_{1j}$$

$$= \frac{1}{\Sigma_{11}} \sum_{j=1}^{p} \Sigma_{1j}\sigma_{1j}$$

$$= \frac{|\Sigma|}{\Sigma_{11}}$$

$$= \frac{\sigma_{11}\sigma_{22}\cdots\sigma_{pp}|R|}{\sigma_{22}\cdots\sigma_{pp}R_{11}}$$

$$= \frac{\sigma_{11}|R|}{R_{11}}$$

$$\boxed{V(e) = Cov(X_1, e) = \frac{|\Sigma|}{\Sigma_{11}} = \frac{\sigma_{11}|R|}{R_{11}}}$$

(iv) $Corr(X_1, e) = \dfrac{Cov(X_1, e)}{\sqrt{Var(X_1)}\sqrt{Var(e)}}$

$$= \frac{\sigma_{11}|R|}{R_{11}} \cdot \frac{\sqrt{R_{11}}}{\sqrt{\sigma_{11}|R|}} \cdot \frac{1}{\sqrt{\sigma_{11}}}$$

$$\boxed{Corr(X_1, e) = \sqrt{\frac{|R|}{R_{11}}}}$$

Partial and Multiple correlation in p-variate Normal distribution

—— x —— x —— x —— x

Partial and multiple correlation in p-variate Normal Distⁿ :-

Measure of dependence b/w two normal variates is called correlation coeff. and is given by $\rho_{ij} = \dfrac{\sigma_{ij}}{\sqrt{\sigma_{ii}}\sqrt{\sigma_{jj}}}$

In a conditional distⁿ of $[x_1 x_2 \ldots x_q | x_{q+1} \ldots x_p]$, the partial correlation $\rho_{ij \cdot q+1, \ldots p}$ measure the dependence b/w x_i's and x_j's.

The 3rd kind of correlation to be discussed, is the multiple correlation which measure the relationship b/w 1 variate and the set of others.

Let us consider the problem of measuring the degree of which one of the vari. say x_1 may be said to be dependant on $x_2, \ldots x_p$ taking jointly in order to measure the degree of coeff. we use multiple correlation coeff. x_1 on $x_2, \ldots x_p$.

Let $\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$ and we partition $\underset{\sim}{X}, \underset{\sim}{\mu}$ & $\underset{\sim}{\Sigma}$ as follows :-

$$\underset{\sim}{X} = \begin{bmatrix} \underset{\sim}{X}^{(1)} \\ \underset{\sim}{X}^{(2)} \end{bmatrix}, \quad \underset{\sim}{\mu} = \begin{bmatrix} \underset{\sim}{\mu}^{(1)} \\ \underset{\sim}{\mu}^{(2)} \end{bmatrix} \text{ \& } \underset{\sim}{\Sigma} = \begin{bmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}_{21} & \underset{\sim}{\Sigma}_{22} \end{bmatrix}$$

wkt $\underset{\sim}{X}^{(1)} | \underset{\sim}{X}^{(2)} \sim N_q\left[\underset{\sim}{\mu}^{(1)} + \underset{\sim}{\Sigma}_{12} \underset{\sim}{\Sigma}_{22}^{-1}(\underset{\sim}{X}^{(2)} - \underset{\sim}{\mu}^{(2)}), \underset{\sim}{\Sigma}_{11 \cdot 2}\right]$

conditional distⁿ of $\underset{\sim}{\Sigma}_{11 \cdot 2} = \underset{\sim}{\Sigma}_{11} - \underset{\sim}{\Sigma}_{12} \underset{\sim}{\Sigma}_{22}^{-1} \underset{\sim}{\Sigma}_{21}$

The mean vector of conditional distⁿ is called the regression of X_1 on X_2 and the matrix $\Sigma_{12} \Sigma_{22}^{-1} = \beta$ (say) is known as regression coeff. matrix of $X^{(1)}$ on $X^{(2)}$

The (i^{th}, j^{th}) element of matrix $\Sigma_{11\cdot 2}$ is denoted by $\sigma_{ij\cdot(q+1)\cdots p}$ or $\sigma_{ij\cdot r}$ where $r = (q+1)\cdots p$

$$\Sigma_{11\cdot 2} = \begin{bmatrix} \sigma_{11\cdot r} & \sigma_{12\cdot r} & \cdots & \sigma_{1q\cdot r} \\ \sigma_{21\cdot r} & \sigma_{22\cdot r} & \cdots & \sigma_{2q\cdot r} \\ \vdots & \vdots & & \vdots \\ \sigma_{q1\cdot r} & \sigma_{q2\cdot r} & \cdots & \sigma_{qq\cdot r} \end{bmatrix}$$

$$\Sigma_{12} = \begin{bmatrix} \sigma_{1(q+1)} & \sigma_{1(q+2)} & \cdots & \sigma_{1p} \\ \sigma_{2(q+1)} & \sigma_{2(q+2)} & \cdots & \sigma_{2p} \\ \vdots & & & \\ \sigma_{i(q+1)} & \sigma_{i(q+2)} & \cdots & \sigma_{ip} \\ \vdots & & & \\ \sigma_{q(q+1)} & \sigma_{q(q+2)} & \cdots & \sigma_{qp} \end{bmatrix}_{q\times(p-q)}$$

$$= \begin{bmatrix} \sigma_{(1)}' \\ \sigma_{(2)}' \\ \vdots \\ \sigma_{(i)}' \\ \vdots \\ \sigma_{(q)}' \end{bmatrix}$$

$\therefore$ i^{th} row $\Sigma_{12} \Sigma_{22}^{-1} = \beta$ (say)

$= \sigma_i' \Sigma_{22}^{-1} = \beta_{(i)}'$ (say)

where β is the matrix of regression coeff. of $X^{(1)}$ on $X^{(2)}$.

Multiple Correlation :- Let X be a p-dimentional vector and distributed accordingly $N_p(\mu, \Sigma)$ and we partition into $X^{(1)}$ and $X^{(2)}$ with q and $p-q$ component respectively. Let us denote $\beta = \Sigma_{12}\Sigma_{22}^{-1}$ and $\sigma_{(i)}'$ be the i^{th} row of Σ_{12} and $\beta_{(i)}'$ is the i^{th} row of β i.e. $\beta_{(i)}' = \sigma_{(i)}' \Sigma_{22}^{-1}$. And let α be any other vector.

Then maximum correlation b/w X_i; $i = 1, 2, \cdots q$ and $\alpha' X^{(2)}$ is called multiple correlation coeff. b/w X_i and $X^{(2)}$. We know that, the corr. b/w X_i and the linear combination $\alpha' X^{(2)}$ is maximum when $\alpha = \beta_{(i)}$. Thus, by def. Multiple corr. coeff. b/w X_i and $X^{(2)}$ is defined as

$$R_{i\cdot(q+1)\cdots p} = \frac{E[X_i, \beta_{(i)}' X^{(2)}]}{\sqrt{\sigma_{ii}}\sqrt{E\{\beta_{(i)}' X^{(2)}\}^2}}$$

$$= \frac{\beta_{(i)}' E(X_i X^{(2)})}{\sqrt{\sigma_{ii}}\sqrt{\beta_{(i)}' E(X^{(2)} X^{(2)\prime}) \beta_{(i)}}}$$

$$= \frac{\sigma_{(i)}' \Sigma_{22}^{-1} \sigma_{(i)}}{\sqrt{\sigma_{ii}}\sqrt{\sigma_{(i)}' \cancel{\Sigma_{22}^{-1}} \cancel{\Sigma_{22}} \Sigma_{22}^{-1} \sigma_{(i)}}}$$

$$= \frac{\sigma_{(i)}' \Sigma_{22}^{-1} \sigma_{(i)}}{\sqrt{\sigma_{ii}}\sqrt{\sigma_{(i)}' \Sigma_{22}^{-1} \sigma_{(i)}}}$$

$$= \sqrt{\frac{\sigma_{(i)}' \Sigma_{22}^{-1} \sigma_{(i)}}{\sigma_{ii}}}$$

(13) **Estimation of MCC (multiple correlation coeff):-**

The pop^n MCC b/w x_1 and $x_2, x_3, \ldots, x_p$ is

$$\bar{R}_{1.23\ldots p} = \sqrt{\frac{\sigma'_{(1)} \Sigma_{22}^{-1} \sigma_{(1)}}{\sigma_{11}}}$$

$$= \sqrt{\frac{(\sigma'_{(1)} \Sigma_{22}^{-1}) \Sigma_{22} (\Sigma_{22}^{-1} \sigma_{(1)})}{\sigma_{11}}}$$

$$= \sqrt{\frac{\beta'_{(1)} \Sigma_{22} \beta_{(1)}}{\sigma_{11}}} \qquad \left\{ \beta'_{(1)} = \sigma'_{(1)} \Sigma_{22}^{-1} \right.$$

where σ_{11} and Σ_{22} are the elements of

$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma'_{(1)} \\ \sigma_{(1)} & \Sigma_{22} \end{bmatrix}$ given a sample $x_\alpha; \alpha = 1, 2, \ldots N; N > p$ from $N_p(\mu, \Sigma)$. The MLE of Σ is $\frac{A}{N}$. where, $A = \sum_{\alpha=1}^{N} (x_\alpha - \bar{x})(x_\alpha - \bar{x})'$

$$A = \begin{bmatrix} a_{11} & a'_{(1)} \\ a_{(1)} & A_{22} \end{bmatrix}$$

$$\hat{\Sigma} = \frac{A}{N} = \begin{bmatrix} a_{11}/N & a'_{(1)}/N \\ a_{(1)}/N & A_{22}/N \end{bmatrix} = \begin{bmatrix} \hat{\sigma}_{11} & \hat{\sigma}'_{(1)} \\ \hat{\sigma}_{(1)} & \hat{\Sigma}_{22} \end{bmatrix}$$

The estimate of β' is

$$\hat{\beta}' = \hat{\sigma}'_{(1)} \hat{\Sigma}_{22}^{-1} = \frac{a'_{(1)}}{N} \left(\frac{A_{22}}{N} \right)^{-1}$$

$$\hat{\beta}' = a'_{(1)} A_{22}^{-1}$$

Using the above estimates the sample MCC b/w x_1 and $x_2, x_3, \ldots x_p$ is (14)

$$R_{1.23\ldots p} = \sqrt{\frac{\hat{\sigma}'_{(1)} \hat{\Sigma}_{22}^{-1} \hat{\sigma}_{(1)}}{\hat{\sigma}_{11}}} = \sqrt{\frac{a'_{(1)} A_{22}^{-1} a_{(1)}/N}{a_{11}/N}}$$

$$R_{1.23\ldots p} = \sqrt{\frac{a'_{(1)} A_{22}^{-1} a_{(1)}}{a_{11}}}$$

Dist^n of Sample MCC in Null Case (When the pop^n corr coeff. is zero):-

Let $R = R_{1.23\ldots p}$ be the sample MCC b/w x_1 and $x_2, x_3, \ldots x_p$ based on sample of size N from $N_p(\mu, \Sigma)$. If $R = R_{1.23\ldots p} = 0$ then

$$\frac{R^2}{1-R^2} \cdot \frac{N-p}{p-1} \sim F_{p-1, N-p}$$

Proof:- The sample MCC b/w x_1 and $x_2, x_3 \ldots x_p$ is defined as

$$R^2 = \frac{a'_{(1)} A_{22}^{-1} a_{(1)}}{a_{11}}$$

$$1 - R^2 = \frac{a_{11} - a'_{(1)} A_{22}^{-1} a_{(1)}}{a_{11}} = \frac{a_{11.2}}{a_{11}} \text{ (say)}$$

and $$\frac{R^2}{1-R^2} = \frac{a'_{(1)} A_{22}^{-1} a_{(1)}}{a_{11.2}}$$

WKT $\frac{a_{11.2}}{\sigma_{11.2}}$ and $\frac{a'_{(1)} A_{22}^{-1} a_{(1)}}{\sigma_{11.2}}$ are ind chi-square variate with $N-p$ and $p-1$ df.

Therefore,

$$\frac{R^2}{1-R^2}\cdot\frac{(N-p)}{(p-1)} = \frac{(a_{(1)}' A_{22}^{-1} a_{(1)}/\sigma_{11\cdot 2})/(p-1)}{(a_{11\cdot 2}/\sigma_{11\cdot 2})/(N-p)}$$

$$= \frac{\chi^2_{p-1}/(p-1)}{\chi^2_{N-p}/N-p} \sim F_{p-1, N-p}$$

$\therefore$ the distⁿ of F

$$F = \frac{R^2}{1-R^2}\cdot\frac{(N-p)}{(p-1)}$$

$$dP(F) = \frac{1}{\beta\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)} \frac{(\nu_1/\nu_2)^{\nu_1/2}\cdot F^{\frac{\nu_1}{2}-1}}{\left(1+\frac{\nu_1}{\nu_2}F\right)^{\frac{\nu_1+\nu_2}{2}}}\, dF$$

where, $\nu_1 = p-1$ and $\nu_2 = N-p$ and $0 < F < \infty$

Now,

$$F = \frac{R^2}{1-R^2}\cdot\frac{N-p}{p-1}$$

$$= \frac{R^2}{1-R^2}\cdot\frac{\nu_2}{\nu_1}$$

$$dF = \frac{\nu_2}{\nu_1}\cdot\frac{1}{(1-R^2)^2}\, dR^2$$

and

$$1+\frac{\nu_1}{\nu_2}F = \frac{1}{1-R^2}$$

$$dP(R^2) = \frac{(\nu_1/\nu_2)^{\nu_1/2}}{\beta\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)\left(\frac{1}{1-R^2}\right)^{\frac{\nu_1+\nu_2}{2}}}\left[\frac{R^2}{1-R^2}\cdot\frac{\nu_2}{\nu_1}\right]^{\frac{\nu_1}{2}-1}\cdot\frac{\nu_2}{\nu_1}\cdot\frac{1}{(1-R^2)^2}\, dR^2$$

$$= \frac{1}{\beta\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)}(R^2)^{\frac{\nu_1}{2}-1}(1-R^2)^{\frac{\nu_2}{2}+1-2}\cdot 2R\, dR \quad ; -1 < R < 1$$

$$= \frac{(R^2)^{\frac{\nu_1}{2}-1}(1-R^2)^{\frac{\nu_2}{2}-1}}{\beta\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)} \quad ; 0 < R < 1$$

** If we put $p=2$, $\nu_1 = 1$, $\nu_2 = N-2$, we get the distⁿ of ordinary sample corr. coeff.

Thm:- Let $\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \Sigma)$ then for all the linear combinations of type $\alpha'\underset{\sim}{X}^{(2)}$ that minimize the variation of $(X_i - \underset{\sim}{\alpha}'\underset{\sim}{X}^{(2)})$ and maximize the corr. b/w X_i and $\underset{\sim}{\alpha}'\underset{\sim}{X}^{(2)}$ is the linear combination $\beta_{(i)}'\underset{\sim}{X}^{(2)}$ i.e. $\alpha' = \beta_{(i)}'$ being the regression coeff. vector and $X_i \sim X^{(1)}$ $\forall\ i = 1, 2, \ldots q$. Here, $\beta_{(i)}'$

Proof:- Since, we are interested only in functions of the cov. we shall assume without loss of generality that

$$E(\underset{\sim}{X}) = \underset{\sim}{\mu} = 0$$

Now, let us consider the cov. b/w $\underset{\sim}{X}^{(2)}$ and $(X_i - \beta_{(i)}'\underset{\sim}{X}^{(2)})$ i.e.

$$\text{Cov}\left((X_i - \beta_{(i)}'\underset{\sim}{X}^{(2)}), \underset{\sim}{X}^{(2)}\right)$$

$$= E\left[(X_i - \beta_{(i)}'\underset{\sim}{X}^{(2)})(\underset{\sim}{X}^{(2)})'\right]$$

$$= E\left[X_i \underset{\sim}{X}^{(2)'}\right] - \beta_{(i)}' E\left[\underset{\sim}{X}^{(2)}\underset{\sim}{X}^{(2)'}\right]$$

$$= E\left[\{X_i \cdot X_{q+1}, X_i X_{q+2}, \ldots X_i X_p\}\right] - \beta_{(i)}' \Sigma_{22}$$

$= \sigma_{i(q+1)} \; \sigma_{i(q+2)} \cdots \sigma_{i(p)} - \beta'_{(i)} \Sigma_{22}$

$= \sigma'_{(i)} - \sigma'_{(i)} \Sigma_{22}^{-1} \Sigma_{22}$

$= 0$ ——(1)

Now, we are to find α' such that $Var(X_i - \alpha' X^{(2)})$ is min. Consider,

$V(X_i - \alpha' X^{(2)}) = E[(X_i - \alpha' X^{(2)})]^2$

$= E[X_i - \beta'_{(i)} X^{(2)} + \beta'_{(i)} X^{(2)} - \alpha' X^{(2)}]^2$

$= E[X_i - \beta'_{(i)} X^{(2)}]^2 + E[(\beta'_{(i)} X^{(2)} - \alpha' X^{(2)})]^2$

$= E(X_i)^2 + \beta'_{(i)} V(X^{(2)}) \beta_{(i)} - 2\beta'_{(i)} E[X_i X^{(2)}] + (\beta'_{(i)} - \alpha') V(X^{(2)}) (\beta'_{(i)} - \alpha')'$

$= \sigma_{ii} + \beta'_{(i)} \Sigma_{22} \Sigma_{22}^{-1} \sigma_{(i)} - 2\beta'_{(i)} \sigma_{(i)} + (\beta'_{(i)} - \alpha') \Sigma_{22} (\beta'_{(i)} - \alpha')'$

$= \sigma_{ii} - \beta'_{(i)} \sigma_{(i)} + (\beta'_{(i)} - \alpha') \Sigma_{22} (\beta'_{(i)} - \alpha')'$

Thus, $V(X_i - \alpha' X^{(2)})$ is min. when $\beta'_{(i)} = \alpha'$.

$\therefore \quad E(X_i - \beta'_{(i)} X^{(2)})^2 \leq E(X_i - c\alpha' X^{(2)})^2$ —— (2)

where, $c\alpha'$ be any other linear combination.

Now, from expression of the above inequality, we have

$E(X_i)^2 + E[\beta'_{(i)} X^{(2)}]^2 - 2E(X_i \beta'_{(i)} X^{(2)}) \leq E(X_i)^2 + E(c\alpha' X^{(2)})^2 - 2cE(X_i \alpha' X^{(2)})$

Multiply both sides by "−" sign, we get

$-E[\beta'_{(i)} X^{(2)}]^2 + 2E[X_i \beta'_{(i)} X^{(2)}] \geq -c^2 E(\alpha' X^{(2)})^2 + 2cE(X_i \alpha' X^{(2)})$

On dividing the above inequality by $\sqrt{\sigma_{ii} E(\beta'_{(i)} X^{(2)})^2}$ we have

$$\frac{-E[\beta'_{(i)} X^{(2)}]^2 + 2E[X_i \beta'_{(i)} X^{(2)}]}{\sqrt{\sigma_{ii} E(\beta'_{(i)} X^{(2)})^2}} \geq \frac{-c^2 E(\alpha' X^{(2)})^2 + 2cE(X_i \alpha' X^{(2)})}{\sqrt{\sigma_{ii} E(\beta'_{(i)} X^{(2)})^2}}$$

Now choosing $c^2 = \dfrac{E(\beta'_{(i)} X^{(2)})^2}{E(\alpha' X^{(2)})^2}$, we have,

$$\frac{-E(\beta'_{(i)} X^{(2)})^2}{\sqrt{\sigma_{ii} E(\beta'_{(i)} X^{(2)})^2}} + \frac{2E(X_i \beta'_{(i)} X^{(2)})}{\sqrt{\sigma_{ii} E(\beta'_{(i)} X^{(2)})^2}} \geq \frac{-E(\beta'_{(i)} X^{(2)})^2}{\sqrt{\sigma_{ii} E(\beta'_{(i)} X^{(2)})^2}} + \frac{2cE(X_i \alpha' X^{(2)})}{\sqrt{\sigma_{ii} E(\beta'_{(i)} X^{(2)})^2}}$$

$$\Rightarrow \frac{E(X_i \beta'_{(i)} X^{(2)})}{\sqrt{\sigma_{ii} E(\beta'_{(i)} X^{(2)})^2}} \geq \frac{cE(X_i \alpha' X^{(2)})}{\sqrt{\sigma_{ii} E(\beta'_{(i)} X^{(2)})^2}}$$

$$\Rightarrow \frac{E(x_i \beta'_{(i)} X^{(2)})}{\sqrt{\sigma_{ii} E(\beta'_{(i)} X^{(2)})^2}} \geq \frac{E[x_i c\alpha' X^{(2)}]}{\sqrt{\sigma_{ii} E(\beta'_{(i)} X^{(2)})^2}}$$

which is true only for all l.c. $c\alpha'$; Hence, corr. is ~~maximum~~ ~~minimum~~ maximum at $\alpha' = \beta'_{(i)}$. Proved.

Partial Correlation :- Let us consider a p-component vector

$X \sim N_p(\mu, \Sigma)$ where, X, μ and Σ are partitions as follows:

$$X = \begin{bmatrix} X^{(1)} \\ X^{(2)} \end{bmatrix}, \quad \mu = \begin{bmatrix} \mu^{(1)} \\ \mu^{(2)} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

WKT the conditional distn of

$$X^{(1)} | X^{(2)} \sim N_q(\mu^{(1)} + \beta(X^{(2)} - \mu^{(2)});\ \Sigma_{11\cdot 2})$$

where, $\beta = \Sigma_{12}\Sigma_{22}^{-1}$ and

$$\Sigma_{11\cdot 2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

Let $\sigma_{ij\cdot q+1, \dots p}$ be the $(i,j)^{th}$ element of $\Sigma_{11\cdot 2}$ and it is known as partial covariance.

Partial corr. coeff. b/w $x_i \in X^{(1)}$ and $x_j \in X^{(2)}$ holding $x_{q+1}\, x_{q+2} \dots x_p$ as fixed is given by

$$\rho_{ij\cdot(q+1)\dots p} = \frac{\sigma_{ij\cdot(q+1)\dots p}}{\sqrt{\sigma_{ii(q+1)\dots p} \cdot \sigma_{jj\cdot(q+1)\dots p}}}$$

Estimation of Partial Correlation Coefficient :-

Thm :- Let $X_\alpha \sim N_p(\mu, \Sigma)$, $\alpha = 1, 2, \dots N$; $N > p$ be a sample of size N. Then the MLE of partial corr. coeff. are given by

$$\hat{\rho}_{ij\cdot(q+1)\dots p} = \frac{a_{ij\cdot(q+1)\dots p}}{\sqrt{a_{ii\cdot(q+1)\dots p} \cdot a_{jj\cdot(q+1)\dots p}}}$$

where, $a_{ij\cdot(q+1)\dots p}$ be the $(i,j)^{th}$ element of matrix

$$A_{11\cdot 2} = A_{11} - A_{12}A_{22}^{-1}A_{21}$$

Proof :- If we take a sample of size N from $N_p(\mu, \Sigma)$, then the MLE of Σ is given by $\frac{A}{N}$

i.e. $$\hat{\Sigma} = \frac{A}{N} = \frac{1}{N}\sum_{\alpha=1}^{N}(X_\alpha - \bar{X})(X_\alpha - \bar{X})'$$

Then MLE of $\Sigma_{11\cdot 2}$ is given by

$$\hat{\Sigma}_{11\cdot 2} = \hat{\Sigma}_{11} - \hat{\Sigma}_{12}\hat{\Sigma}_{22}^{-1}\hat{\Sigma}_{21}$$

where, $$\hat{\Sigma} = \begin{bmatrix} \hat{\Sigma}_{11} & \hat{\Sigma}_{12} \\ \hat{\Sigma}_{21} & \hat{\Sigma}_{22} \end{bmatrix} = \frac{1}{N}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$\hat{\Sigma}_{11\cdot 2} = \frac{A_{11}}{N} - \frac{A_{12}}{N} \cdot \frac{A_{22}^{-1}}{(N)^{-1}} \cdot \frac{A_{21}}{N}$$

$$= \frac{1}{N}(A_{11\cdot 2})$$

If we denote the $(i,i)^{th}$ and $(i,j)^{th}$ element of $\hat{\Sigma}_{11\cdot 2}$

$\hat{\sigma}_{ii} = \frac{1}{N}\left(a_{ii\cdot(q+1)---p}\right)$

$\hat{\sigma}_{ij} = \frac{1}{N}\left(a_{ij\cdot(q+1)---p}\right)$

Then the transformation from $\Sigma_{11\cdot2}$ to $\hat{\Sigma}_{11\cdot2}$ is one two one. Hence the MLE of $\rho_{ij\cdot(q+1)---p}$ is given by

$$\hat{\rho}_{ij\cdot(q+1)---p} = \frac{a_{ij\cdot(q+1)---p}}{\sqrt{a_{ii\cdot(q+1)---p}\cdot a_{jj\cdot(q+1)---p}}}$$

The estimate of $\rho_{ij\cdot(q+1)--p}$ is denoted by $r_{ij\cdot(q+1)---p}$ and is known as Sample Partial Corr. Coeff. b/w x_i and x_j holding $x_{(q+1)---p}$ fixed.

Theorem :- The matrix $A_{11\cdot2} = A_{11} - A_{12}A_{22}^{-1}A_{21}$ is distributed as $\sum_{\alpha=1}^{(N-1)-p-q} U_\alpha U_\alpha'$ where, $U_\alpha \sim N_q(0, \Sigma_{11\cdot2})$

Also if $\Sigma_{12} = 0$, then $A_{12}A_{22}^{-1}A_{21}$ is distributed as $\sum_{\alpha=N-(p-q)}^{N-1} U_\alpha U_\alpha'$

Proof :- We know that,

$$A = \sum_{\alpha=1}^{N-1} Z_\alpha Z_\alpha' \quad ; \quad Z_\alpha \sim N_p(0, \Sigma)$$

If we partition $Z_\alpha = \begin{bmatrix} Z_\alpha^{(1)} \\ Z_\alpha^{(2)} \end{bmatrix}$, Then we get

$$A = \begin{bmatrix} \sum_{\alpha=1}^{N-1} Z_\alpha^{(1)} Z_\alpha^{(1)'} & \sum_{\alpha=1}^{N-1} Z_\alpha^{(1)} Z_\alpha^{(2)'} \\ \sum_{\alpha=1}^{N-1} Z_\alpha^{(2)} Z_\alpha^{(1)'} & \sum_{\alpha=1}^{N-1} Z_\alpha^{(2)} Z_\alpha^{(2)'} \end{bmatrix}$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Also, we know that the conditional density of $Z_\alpha^{(1)} \mid Z_\alpha^{(2)} = z_\alpha^{(2)}$; $\alpha = 1, 2, --(N-1)$ is given by

$$Z_\alpha^{(1)} \mid Z_\alpha^{(2)} = z_\alpha^{(2)} \sim \prod_{\alpha=1}^{N-1} N_q\left(z_\alpha^{(1)} \mid \beta z_\alpha^{(2)}, \Sigma_{11\cdot2}\right)$$

where, $\beta = \Sigma_{12}\Sigma_{22}^{-1}$

Now, we use the following thm :

Statement :- Suppose $Y_\alpha \sim N_p(T w_\alpha, \phi)$, $\alpha = 1, 2, -m$, where w_α is the r component vector, $m > r$, and T is a matrix of $p \times r$. Let

$$G = \sum_{\alpha=1}^{m} Y_\alpha w_\alpha' H^{-1}$$

where $H = \sum_{\alpha=1}^{m} w_\alpha w_\alpha'$ is the non-singular matrix then $\sum_{\alpha=1}^{m} Y_\alpha Y_\alpha' - GHG' \sim \sum_{\alpha=1}^{m-r} U_\alpha U_\alpha'$ —(1)

(23) where, $U_\alpha \sim N_{p-r}(0, \phi)$, also ind of $\{Y_\alpha\}$

Now in the above thm put the following notations

$$w_\alpha = z_\alpha^{(2)}, \; m = N-1, \; r = p-2, \; T = \beta,$$

$$\phi = \Sigma_{11\cdot 2}, \; Y_\alpha = Z_\alpha^{(1)}$$

$$\sum_{\alpha=1}^{m} Y_\alpha Y_\alpha' = A_{11}, \; G = A_{12}A_{22}^{-1}, \; H = A_{22}$$

Then eq^n (1) gives

$A_{11\cdot 2} = A_{11} - A_{12}A_{22}^{-1}A_{21}$ is distributed as

$$A_{11\cdot 2} = A_{11} - A_{12}A_{22}^{-1}A_{21} \sim \sum_{\alpha=1}^{(N-1)-(p-2)} U_\alpha U_\alpha' \text{ where,}$$

$U_\alpha \sim N_q(0, \Sigma_{11\cdot 2})$

Note that the dist^n is free from $Z_\alpha^{(2)}$ and hence is also unconditional dist^n. Now, since

$$A_{11} - A_{12}A_{22}^{-1}A_{21} = \sum_{\alpha=1}^{(N-1)-(p-2)} U_\alpha U_\alpha'$$

$$A_{12}A_{22}^{-1}A_{21} = \sum_{\alpha=1}^{N-1} U_\alpha U_\alpha' - \sum_{\alpha=1}^{(N-1)-(p-2)} U_\alpha U_\alpha'$$

$$= \sum_{\alpha=N-(p-2)}^{N-1} U_\alpha U_\alpha' \qquad \left\{ \begin{array}{l} \Sigma_{11\cdot 2} = \Sigma_{11} \\ \because \Sigma_{12} = 0 \end{array} \right.$$

$\alpha = N-(p-2)$
$= N-(p-2)$

where, $U_\alpha \sim N_q(0, \Sigma_{11})$

<u>Proved.</u>

Sampling Dist^n of Partial Corr. Coeff. when (24)
pop^n corr. coeff is zero. i.e. $\rho = 0$ (or Null case)

For the convinience we will derive the dist^n of r_{12} instead of r_{ij} as follows, we have

$$r = r_{12} = \frac{a_{12}}{\sqrt{a_{11}a_{22}}} = \frac{\sum_{\alpha=1}^{N-1}(x_{1\alpha}-\bar{x}_1)(x_{2\alpha}-\bar{x}_2)}{\sqrt{\sum_{\alpha=1}^{N-1}(x_{1\alpha}-\bar{x}_1)^2 \sum_{\alpha=1}^{N-1}(x_{2\alpha}-\bar{x}_2)^2}}$$

$$= \frac{\sum_{\alpha=1}^{N-1}(x_{1\alpha}-\bar{x}_1)x_{2\alpha} - \bar{x}_2 \sum_{\alpha=1}^{N-1}(x_{1\alpha}-\bar{x}_1)}{n s_1 s_2} \; ; n = N-1$$

$$= \frac{\sum_{\alpha=1}^{N-1}(x_{1\alpha}-\bar{x}_1)x_{2\alpha}}{n s_1 s_2} \qquad \left\{ \because \sum_{\alpha=1}^{N-1}(x_{1\alpha}-\bar{x}_1) = 0 \right.$$

$$\Rightarrow \sqrt{n}\, s_2 r = \frac{\sum_{\alpha=1}^{n}(x_{1\alpha}-\bar{x}_1)x_{2\alpha}}{\sqrt{n}\, s_1} \quad \text{--- (1)}$$

Now without loss of generality we assume that $E(x_1) = E(x_2) = 0$. And $E(x_1)^2 = V(x_1) = \sigma_{11}$,

$E(x_2)^2 = V(x_2) = \sigma_{22}$.

Let us consider an orthogonal transformation

$$Z_\alpha = \sum_{\beta=1}^{N} b_{\alpha\beta} x_{2\beta} \quad \text{--- (2)} \quad \alpha = 1, 2, \ldots N$$

where, $B = b_{\alpha\beta}$ with its first row as matrix $\quad Z_1 = \sum_{\beta=1}^{N} b_{1\beta} x_{2\beta}$

$\underset{\sim}{B} = b_{\alpha\beta} = \left[\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}, \ldots \frac{1}{\sqrt{N}}\right]$ and its second row as

$$\left[\frac{(x_{11}-\bar{x}_1)}{S_1\sqrt{n}}, \frac{(x_{12}-\bar{x}_1)}{S_1\sqrt{n}}, \ldots \frac{(x_{1n}-\bar{x}_1)}{\sqrt{n}S_1}\right] \text{ then,}$$

from eq^n (2), we have

$$Z_1 = \frac{1}{\sqrt{N}}\left[x_{21}+x_{22}+\cdots+x_{2N}\right] = \frac{1}{\sqrt{N}}\sum x_{2\beta}$$

$$= \frac{N\bar{x}_2}{\sqrt{N}}$$

$\Rightarrow$

$$Z_1 = \sqrt{N}\,\bar{x}_2 \quad \text{--- (3)}$$

And $$Z_2 = \frac{\sum_{\beta=1}^{n}(x_{1\beta}-\bar{x}_1)(x_{2\beta})}{\sqrt{n}S_1}$$

$$= \sqrt{n}S_2 r \; (\text{from eq}^n\ (1)) \quad \text{--- (4)}$$

Now, we have

$$E(Z_\alpha) = 0 \quad \forall\, \alpha = 1, 2, \ldots N$$

and $$V(Z_\alpha) = \sum_{\beta=1}^{N} b_{\alpha\beta} b'_{\alpha\beta} V(x_2)$$

$$= \sigma_{22} \quad \{\because \underset{\sim}{B} \text{ is an orthogonal matrix}\}$$

$$\therefore \quad Z_\alpha \sim N(0, \sigma_{22})$$

Further we have from eq^n (2)

$$\sum_{\alpha=1}^{N} Z_\alpha^2 = \sum_{\alpha=1}^{N} x_{2\alpha}^2 = \sum_{\alpha=1}^{N}(x_{2\alpha}-\bar{x}_2+\bar{x}_2)^2$$

$$\Rightarrow Z_1^2 + Z_2^2 + \sum_{\alpha=3}^{N} Z_\alpha^2 = \sum_{\alpha=1}^{N}(x_{2\alpha}-\bar{x}_2)^2 + N\bar{x}_2^2 + 0$$

$$\Rightarrow Z_1^2 + Z_2^2 + \sum_{\alpha=3}^{N} Z_\alpha^2 = nS_2^2 + N\bar{x}_2^2$$

$$\Rightarrow \sum_{\alpha=3}^{N} Z_\alpha^2 = nS_2^2 + \cancel{N\bar{x}_2^2} - \cancel{N\bar{x}_2^2} - nS_2^2 r^2$$

$$= (1-r^2)nS_2^2 \quad \text{--- (5)}$$

As we have,

$$Z_\alpha \sim N(0, \sigma_2^2) \quad \forall\, \alpha = 1, 2, \ldots N$$

$$\frac{Z_\alpha^2}{\sigma_2^2} \sim \chi_1^2 \text{ df}$$

$\therefore$ from eq^n (4), we have

$$U = \frac{Z_2^2}{\sigma_2^2} = \frac{nS_2^2 r^2}{\sigma_2^2} \sim \chi_1^2 \text{ df} \quad \text{--- (6)}$$

And from eq^n (5)

$$V = \frac{\sum_{\alpha=3}^{N} Z_\alpha^2}{\sigma_2^2} = \frac{(1-r^2)nS_2^2}{\sigma_2^2} \sim \chi^2_{(N-2)} \text{ df}$$

Now, $$r^2 = \frac{nS_2^2 r^2/\sigma_2^2}{\left[nS_2^2 r^2 + (1-r^2)nS_2^2\right]/\sigma_2^2}$$

$$= \frac{U}{U+V} \sim \frac{\chi_1^2}{\chi_1^2 + \chi_{N-2}^2}$$

$$\Rightarrow r^2 \sim \beta_1\left(\frac{1}{2}, \frac{N-2}{2}\right)$$

which is the sampling dist^n of partial corr. coeff. Proved.

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