Design

of

Experiment

(After 2^2 factorial Experiment)

Orthogonality of a Design and Confounding

Suppose, we have a random sample of n observations $y_1, y_2, \ldots y_n$ which are independent from normal population with variance σ^2.

Let us consider two orthogonal contrast: $A = \sum_{i=1}^{n} \lambda_i y_i$ with $\sum_{i=1}^{n} \lambda_i = 0$

and $B = \sum_{i=1}^{n} \mu_i y_i$ with $\sum_{i=1}^{n} \mu_i = 0$

and $\sum_{i=1}^{n} \lambda_i \mu_i = 0$

Then, we have covariance (AB)

$$Cov(A,B) = E(AB) - E(A)E(B)$$

$$\left\{ \begin{array}{l} E[(\sum_i \lambda_i x_i)(\sum_i \mu_i x_i)] \\ = \sum_{i=1}^{n} \lambda_i \mu_i E(x_i^2) \end{array} \right.$$

$$Cov(A,B) = \sigma^2 \sum_{i=1}^{n} \mu_i \lambda_i = 0$$

Therefore, A and B are independent and we can use A and B to estimate two different effects because the error in two estimate will not be related. These estimates are also said to be orthogonal.

The orthogonality of a design ensures that the different effect will be capable of separate estimation and testing without any difficulties. Hence, if data arise in an orthogonal design then there will no difficulty of independent estimation and test of effects.

The designs C.R.D., R.B.D. and L.S.D give an orthogonal design. The 2^2 and 2^3 etc factorial experiment are conducted by using C.R.D., R.B.D. and L.S.D. However, the difficulty in conducting the factorial experiment in these designs is that as the number of factors ~~or~~ no. of levels of factors or both increases, the no. of treatment combinations to be compared increases rapidly. This results the use of large sized blocks or squares to accomodate all the treatment combinations.

For Example :- In 2^4 factorial experiment there are 16 treatment combinations and it is advisable to adopt a R.B.D. for it. Because blocks of 16 plots are too big, ~~here~~ to ensure homogeneity within blocks, Therefore, it is necessary to have a new device (techniques) for designing experiment with a large ~~of~~ number of treatments or treatment combinations.) Once, such device is to take blocks ~~of~~ size less than the number of treatment combinations and have more than one block per replication. Then, the treatments are divided into as many groups as number of blocks per replication.

The different groups of treatment combination are alotted to blocks in such a way that the only unimportant treatment combinations get mixed up with the block comparison, these treatment combinations are said to be confounded or mixed up with block effects.

These effects can not be estimated for listing separately. However, the remaining treatment effects which are not confounded with the block effect are still capable of separate estimate in testing. In a confounded design we lose information on some treatment combinations (completely or partially) which are confounded. Therefore, their should be least important combination is confounded. Usually we choose highest order interaction for confounding.

The device of confounding consist in subdividing the replicate into two or more equal sub groups (blocks) and the various treatment combinations into these two or more equal groups of a equal size following certain rules by which we sacrifies some information on certain higher order interaction and allocating the treatment combinations of any group to any block at random.

Confounding in factorial Experiment :→

Confounding in factorial experimental design is an arrangement of treatment combinations in blocks in which less important treatment effects are confounded (mixed up) with the blocks.

There are two types of confounding :-

(1) Complete confounding

(2) Partial confounding

when there are two or more replications then the question arises whether we confound some interactions in each replications and different sets of interaction in different replications.

If the same set of interactions is confounded in all the replications then confounding is called complete confounding. In complete confounding all information of confounded interaction is lost and we lost all the information from all the replications.

If different sets of interaction are confounded in different replications then confounding is called partial confounding. In this confounding the confounded interaction can be recovered from those replications in which it is not confounded.

Confounding in a 2^3 experiment :→

In this experiment we have eight treatment combinations under comparission. Suppose, we decide to use two blocks to accomadate eight treatment combinations in a replicate. Thus, we have four plots in each ~~plot~~ block. Now, we are to decide the eight treatment combination allocate into two groups of four treatment combinations in each group at random.

<u>Remark</u> :→ The 2^n factorial experiment conducted using 2^k blocks, we have to confound $2^{n-k}-1$ treatment combinations in which $2^k-(k+1)$ will be generalised and they are automatically confounded

with the blocks effect. If $k=1$ then only one treatment combination will be confounding. If $k>1$ then more than one treatment combination will be confounded according to the above rule.

In this case we have $k=1$ and only one treatment combination will be confounded. Suppose, we decide the highest order interaction ABC to be confounded.

The interaction ABC is depend on

$$(a-1)(b-1)(c-1) = (abc)-(bc)-(ac)+(c)-(ab)+(b)+(a)-(1)$$

Let we apply the four treatment combinations with plus sign in one block and the remaining four treatment combinations with negative sign in second block. Therefore, the block Ist and IInd contain treatment combinations as

+	+	+	+	
abc	a	b	c	→ Block I
1	ab	bc	ac	→ Block II
−	−	−	−	

Here, the contrast measuring the interaction ABC also contain blocks effect i.e.

effect of block first − effect of block second

And we say that the interaction ABC is mixed up or confounded with block effect and we lose information of ABC. However, the other six contrast -

of the treatment combinations A, B, C, AB, BC, CA still maintain their orthogonality to the block i.e. each two treatment combination of block I^st (block II^nd) with plus sign and two treatment combination with negative sign of the remaining six treatment combinations. Therefore, they will contain no block effects and being orthogonal to ABC and blocks.

Thus, in this allocation of eight treatment combinations to two blocks. There is no difficulty in the estimation or testing of the remaining six treatment combinations.

The first two columns of ANOVA Table for 2^3 confounded experiment is given by

ANOVA TABLE

Source of Variation	Degree of freedom	Sum of Squares	Mean Sum of Squares	Variance Ratio
Blocks	$2q-1$	SS(Blocks)	MSS(Blocks)	
A	1	$[A]^2/8q$	MSSA	F = MSSA/MSSE
B	1	$[B]^2/8q$	MSSB	F = MSSB/MSSE
C	1	$[C]^2/8q$	MSSC	F = MSSC/MSSE
AB	1	$[AB]^2/8q$	MSS(AB)	F = MSS(AB)/MSSE
AC	1	$[AC]^2/8q$	MSS(AC)	F = MSS(AC)/MSSE
BC	1	$[BC]^2/8q$	MSS(BC)	$F = \frac{MSS(BC)}{MSSE}$
Error	$6(q-1)$	SSE	MSSE	
Total	$8q-1$			

Comparison of Unconfounded and completely Confounded 2^n experiments :→

We know that the informations of n effects contains in an experiment is the reciprocal of the variance of its estimate. In the case of unconfounded design the replicate is itself a block and in this case we denote the error variance by σ^2. In a completely confounded design the replicate contains two blocks. i.e. A block is a half replicate. Therefore, in this case we denote the error variance by $\sigma^2_{1/2}$ and it is expected that $\sigma^2_{1/2} < \sigma^2$, since the smaller block will have greater control over error as compare to the larger block.

The variance of the estimator of an effect (main effect or interaction effect) of a 2^n ~~effe.~~ experiment in q replicates without confounding is $\frac{\sigma^2}{q2^{n-2}}$ whereas the variance of the estimator of each unconfounded effect in a 2^n experiment in q replicates (where highest order interaction is completely confounded) is $\frac{\sigma^2_{1/2}}{q2^{n-2}}$. Thus, the information about each effect in an unconfounded design is $\frac{q2^{n-2}}{\sigma^2}$ whereas the information about each unconfounded effect in a completely confounded design $\frac{q2^{n-2}}{\sigma^2_{1/2}}$.

Since, $\sigma^2_{1/2} < \sigma^2$

Therefore, the confounded designs provides more information of un confounded effect as compared to the unconfounded designs. But the confounded designs provides no information as zero information about the effect has been completely confounded.

Remark → when we are not sure which interactions are unimportant then we cannot sacrifies entire loss of information on that treatment combination. In such cases we distribute the loss of information among more than one treatment combination and we shall get some information on each of them. Then, this can be assume by a partialy confounded design.

Partial Confounding in a 2^3 Experiment :-

Here, we have four ~~treatment~~ interactions AB, BC, AC, ABC. we take four replications and two blocks of size 4 in each replicate. The eight treatment combinations are allotted to a block of a replicate in such a way that the interaction AB is confounded in the first replicate, AC is confounded in the second replicate, BC is confounded in the third replicate and ABC is confounded in the fourth replicate. The layout before randomization will be as follows

Replicate I

B_1	(1)	(ab)	(c)	(abc)
B_2	(a)	(b)	(bc)	(ac)

AB

Replicate II

B_3	(1)	(ac)	(b)	(abc)
B_4	(a)	(c)	(bc)	(ab)

AC

	Replicate III			
B_5	(1)	(bc)	(a)	(abc)
B_6	(b)	(c)	(ac)	(ab)

BC

	Replicate IV			
B_7	(1)	(ab)	(bc)	(ac)
B_8	(a)	(b)	(c)	(abc)

ABC

The first two columns of Anova table in this case will be

S.S.	d.f.
Blocks	7
A	1
B	1
C	1
AB	1
AC	1
BC	1
ABC	1
Error	17
Total	31

The block sum of squares is computed from the eight blocks totals and the Grand total. The sum of squares due to the main effects A, B and C (unconfounded effects) are computed using data from all the four replicates whereas the sum of squares due to any confounded interaction is obtained from those replicates where that particular interaction is not confounded.

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Ques → Given a block how to find the interaction confounded

First of all find the key block. If key block is not given then key block may be obtained by selecting any treatment combination from the given block and multiplying all the treatment combinations in the block by that treatment combination. From the key block we can find the number of factors as well as the block size $(2^5, 2^3)$. Now, find out the unit matrix of order k.

Example →

		A	B	C	D	E
acde	acd	1	0	1	1	0
bcd	bcde	0	1	1	1	1
e	1	0	0	0	0	0
abce	abc	1	1	1	0	0
ad	ade	1	0	0	1	1
bde	bd	0	1	0	1	0
ab	abe	1	1	0	0	1
c	ce	0	0	1	0	1

A	B	C	D	E
1	0	0	1	1
0	1	0	1	0
0	0	1	0	1

$D\,A^1B^1C^0 = ABD$

$E\,A^1B^0C^1 = ACE$

$\Rightarrow BCDE$

it is confounded.

In the similar way we can find out other two unit matrices

General rule for Confounding in 2^n series :→

let the design is $(2^n, 2^k)$. Therefore,

Treatment Combination $= 2^n$

Block Size $= 2^k$

No. of block per replication $= \frac{2^n}{2^k} = 2^{n-k}$

Total no. of interaction confounded $= 2^{n-k} - 1$

No. of independent interaction confounded $= n-k$

Generalised Interaction confounded $= 2^{n-k} - 1 - (n-k)$

Example $\rightarrow$

		A	B	C
a	1	0	0	0
b	ab	1	1	0
ac	c	0	0	1
bc	abc	1	1	1

A	B	C
1	1	0
0	0	1

$A B^1 C^0$.

$\Rightarrow$ AB is confounded.

Imp.

Comparison of the information about un-confounded effect and confounded effect in Partial Confounding Design $\rightarrow$

In partial confounded design the main effects A, B, C are not confounded in any replicate. So, that they are estimated from all four replicates and the experiment contains $8/8 = \frac{2}{2}$ information about each of the main effects. But each interaction is confounded in one replicate and unconfounded in the remaining three repli cates. Thus, we can estimate the **interaction from those replicates where** it is not confounded. For Example → The interaction ABC is confounded from the replicate I, So, this can be estimated using replicates II, III and IV. Only three replicates contains information about the interaction AB. and the amount of the information is $6/\sigma^2_{Y2}$. Thus, relative information of each partially confounded interaction with respect to the unconfounded main effect is

$$\frac{\frac{6}{\sigma^2_{Y2}}}{\frac{8}{\sigma^2_{Y2}}} = \frac{3}{4}$$

and this is the same as the proportion of replicates given information about the confounded interaction.

<u>Table for amount of information about different confounded interaction in 2^3 experiments</u> →

effect	Amount of information		
	unconfounded design	ABC completely confounded	AB, BC, AC, ABC partially confounded
A	$8/\sigma^2$	$8/\sigma^2_{Y2}$	$8/\sigma^2_{Y2}$
B	$8/\sigma^2$	$8/\sigma^2_{Y2}$	$8/\sigma^2_{Y2}$
AB	$8/\sigma^2$	$8/\sigma^2_{Y2}$	$6/\sigma^2_{Y2}$
C	$8/\sigma^2$	$8/\sigma^2_{Y2}$	$8/\sigma^2_{Y2}$
AC	$8/\sigma^2$	$8/\sigma^2_{Y2}$	$6/\sigma^2_{Y2}$
BC	$8/\sigma^2$	$8/\sigma^2_{Y2}$	$6/\sigma^2_{Y2}$
ABC	$8/\sigma^2$	0	$6/\sigma^2_{Y2}$

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Plot size = $\frac{2^n}{2^k} = 2^{n-k}$

Imp

Confounding in 2^4 experiment in 2^2 blocks :-

In this case we have 16 treatment combinations and these treatment combinations are alloted for the blocks in a replicate. The total no. of confounded effects are $2^{4-2} - 1 = 3$ in which $2^2 - 1 - 2 = 1$ will be generalised interaction. Suppose we select ABC and ABD for confounding then

$$ABC\,ABD = A^2B^2CD$$
$$= A^0B^0CD = CD$$

will also be confounded.

we can make first the key blocks of ABC and ABD then we make confounded blocks

The 16 treatment combinations can be written in a systematic order as follows :-

1, a, b, ab, c, ac, bc, abc,

d, ad, bd, abd, cd, acd, bcd, abcd

B_1	B_2	B_3	B_4
(1)	(a)	(c)	(d)
(ab)	(b)	(abc)	(abd)
(acd)	(cd)	(ad)	(ac)
(bcd)	(abcd)	(bd)	(bc)

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Confounding in more than two blocks :-

For 2^4 experiment 2 blocks per replications are reasionable in experiment with a large number of factors. we however use blocks (>2) per replicate confounding with two groups one per replicate confounding more than one in-interaction is per annum, more than one interaction is confounded in such cases the key block contains ~~after~~ are obtained from the solu's more than one homogeneous equations simultaneously.

For Example :- The key block of size 2^3 in 2^5 factorial experiment can be obtained from the solutions of the following equations

$$x_1 + x_2 + x_3 = 0$$
$$x_1 + x_4 + x_5 = 0$$

These equations indicate that the interactions ABC and ADE are confounded simoultaneously in ~~and~~ the same replication. Any solution of the above two homogeneous equations is also a solution of the equation which is obtained from the linear combination of the equation. This shows that some other interaction is also confounded. On adding the above two equations we get $x_2 + x_3 + x_4 + x_5 = 0$

No other equation is possible from their linear combination. Therefore, the interaction BCDE is also confounded which is called generalised interaction of ABC and ADE. The key block can be obtained by first obtaining three independent

Solutions of the homogeneous equations and then taking all linear combination. In this way we get the key block

Key block

0 0 0 0 0	1
0 1 1 1 1	bcde
1 0 1 0 1	ace
1 0 1 1 0	acd
1 1 0 1 0	abd
1 1 0 0 1	abe
0 0 0 1 1	de
0 1 1 0 0	bc

There are three more blocks in a replicate which are obtained from the solutions of the following three sets of equations

$$\left.\begin{array}{l} x_1+x_2+x_3=0 \\ x_1+x_4+x_5=1 \end{array}\right]$$

$$\left.\begin{array}{l} x_1+x_2+x_3=1 \\ x_1+x_4+x_5=0 \end{array}\right]$$

$$\left.\begin{array}{l} x_1+x_2+x_3=1 \\ x_1+x_4+x_5=1 \end{array}\right]$$

3^n Experiments :- when factors are taken at three levels instead of 2 the experiment becomes more informative. let n factors be denoted by A, B, C, - - so on and each having three levels. The levels of three factors are denoted by 0, 1 and 2. The possible combinations of these three levels of each factor will gives arise to 3^n treatment combination. Here, we use the number system reduced modulo 3 i.e.

$0 = 3 = 6 = 9 = \cdots$

$1 = 4 = 7 = 10 =$

$2 = 5 = 8 = 11 =$

In this system we divide a number greater than or equal to 3 by 3 and take the remainder to be equal to the original number.

3^2 - factorial Experiment :-

In this experiment there are two factors (let) A and B each having three levels 0, 1 and 2. Therefore, there are 9 treatment combination of type (x_1, x_2) where, x_1 and x_2 take any of the values 0, 1 and 2. Thus, 9 treatment combinations are (0,0), (1,0), (0,1), (1,1), (2,0), (0,2), (1,2), (2,1), (2,2) or (00, 10, 01, 11, 20, 02, 12, 21, 22). Among these 9 treatment combinations there will be 8 comparisions i.e. degrees of freedom = 8

can be partition as 2 d.f. for each of main effects

A and B and four d.f. for the interaction AB. These components can be obtained by a two-way table as follows

		Level of A			
					Total for B
Level of B	0	00	10	20	$[B]_0$
	1	01	11	21	$[B]_1$
	2	02	12	22	$[B]_2$
Total for A		$[A]_0$	$[A]_1$	$[A]_2$	M

The sum of squares can be obtained as

$$SSA = \frac{[A]_0^2 + [A]_1^2 + [A]_2^2}{3q} - \frac{M^2}{9q}$$

where, q is the number of replicate.

$$SSB = \frac{[B]_0^2 + [B]_1^2 + [B]_2^2}{3q} - \frac{M^2}{9q}$$

$$SS(A \times B) = \frac{[00]^2 + [10]^2 + [20]^2 + [01]^2 + \cdots + [22]^2}{q} - \frac{M^2}{9q}$$

The four degrees of freedom of A×B can be further partition into two more orthogonal contrast each having two degrees of freedom. These components are very useful in confounding in ~~the~~ 3^n experiment. Thus, the interaction A×B carrying four degrees of freedom can be partition as AB and AB^2.

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The defining equations divide the 9 treatment total into three groups, & comparision between which give the corresponding components thus $A \times B$ can be divided as AB and AB^2

$x_1+x_2=0$ gives $[00]+[12]+[21] = [AB]_0$

$x_1+x_2=1$ gives $[01]+[10]+[22]=[AB]_1$

$x_1+x_2=2$ gives $[02]+[20]+[11] = [AB]_2$

$x_1+2x_2=0$ gives $[00]+[11]+[22] = [AB^2]_0$

$x_1+2x_2=1$ gives $[02]+[10]+[21] = [AB^2]_1$

$x_1+2x_2=2$ gives $[01]+[12]+[20] = [AB^2]_2$

Therefore, the sum of squares due to component AB is given by

$$SS(AB) = \frac{[AB]_0^2+[AB]_1^2+[AB]_2^2}{3q} - \frac{M^2}{9q}$$

and

$$SS(AB^2) = \frac{[AB^2]_0^2+[AB^2]_1^2+[AB^2]_2^2}{3q} - \frac{M^2}{9q}$$

The first two columns of anova table are

Source of Variation	Degrees of freedom
Replication	$(q-1)$
A	2
B	2
$A \times B \rightarrow AB$	2
$A \times B \rightarrow AB^2$	2
Error	$8(q-1)$
Total	$9q-1$

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3^3 - Experiment :→

In this experiment we have 27 treatment combinations are of the form (x_1, x_2, x_3) where x_1, x_2 and x_3 are the levels of factor A, B and C respectively. All the treatment ~~can systematically~~ combinations can systematically be written as

000	001	002
100	101	102
010	011	012
110	111	112
200	201	202
210	211	212
020	021	022
120	121	122
220	221	222

The 27 treatment combinations will have a sum of squares carrying 26 degrees of freedom. The treatment sum of squares can be calculated from the 27 treatment totals taking ones q replicates. In this experiment the treatment combinations can be sub divided into the main effects and interaction effects having degree's of freedom as follows

Replicates $-(q-1)$, T.C. $-$ 26,
A $-$ 2, B $-$ 2, C $-$ 2, $A \times B$ $-$ 4,
$A \times C$ $-$ 4, $B \times C$ $-$ 4, $A \times B \times C$ $-$ 8
Error $-$ $26(q-1)$, Total $-$ $(27q-1)$

The sum of squares due to main effect and two factor interaction $A\times B, A\times C, B\times C$ are calculated from the three, two-way tables in the usual manner. However, the three factor interaction sum of squares is obtained by subtraction of these components from the treatment sum of squares.

Here, each set of four or eight degrees of freedom can also be partitions into their orthogonal contrast (components) carrying 2 degrees of freedom as follows

Components of SS		Defining Equation	Mod
$A\times B$	$\to AB$	$x_1+x_2=0,=1,=2$	3
	$\to AB^2$	$x_1+2x_2=0,=1,=2$	3
$A\times C$	$\to AC$	$x_1+x_3=0,=1,=2$	3
	$\to AC^2$	$x_1+2x_3=0,=1,=2$	3
$B\times C$	$\to BC$	$x_2+x_3=0,=1,=2$	3
	$\to BC^2$	$x_2+2x_3=0,=1,=2$	3
ABC	$\to ABC$	$x_1+x_2+x_3=0,=1,=2$	3
	ABC^2	$x_1+x_2+2x_3=0,=1,=2$	3
	AB^2C	$x_1+2x_2+x_3=0,=1,=2$	3
	AB^2C^2	$x_1+2x_2+2x_3=0,=1,=2$	3

Using each of the defining equations we divide the 27 treatment combinations into three groups and a comparision among these three combination totals is the corrs. sum of squares carrying 2 d.f.

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For Example :- The sum of squares due to components AB^2C can be obtained as follows

$$SS(AB^2C) = \frac{[AB^2C]_0^2 + [AB^2C]_1^2 + [AB^2C]_2^2}{9q} - \frac{M^2}{27q}$$

where,

$$[AB^2C]_0 = [000] + [102] + [011] + [110] + \cancel{[102]} + [201] + [212] + [022] + [121] + [220]$$

$$[AB^2C]_1 = [001] + [100] + [012] + [111] + [202] + [210] + [020] + [122] + [221]$$

$$[AB^2C]_2 = [002] + [101] + [010] + [112] + [200] + [211] + [021] + [120] + [222]$$

Similarly the other sum of squares can be obtained by using their defining equations. The each mean sum of squares can be obtained by dividing the corresponding sum of squares by its degrees of freedom.

Anova table first two column

SV	degrees of freedom
Replication	$(q-1)$
T.C.	26
A	2
B	2
$A \times B \to AB$	2
$A \times B \to AB^2$	2
$A \times C \to AC$	2
$A \times C \to AC^2$	2

C	2
B×C	4
A×B×C	8
Error	26(r-1)
Total	(27r-1)

M.Imy

Split Plot Design :- In conducting experiments sometimes some factors have to be applied in larger experimental units while some other factors can be applied in comparatively smaller experimental units.

For Example :- ~~for~~ when, we wish to compare the different method of irrigation and ploughing. In such cases it is possible to introduce a second factor which does not require large plot, with a small number of levels into the same experiment at a little extra cost. This can be done by spliting the plot of the first factor into as many sub plots as there are levels of the second factor.

A split plot design is a design with atleast one blocking factor where the experimental units within each block are assign to the treatment factor levels as usual, and in addition the blocks are assigned at random to the levels of a further treatment factor. The designs have a nested blocking structure. In a split plot design the experimental units are called split plots or sub plots and are nested within the whole plots or main plots.

Layout :- And Randomization :-

There are two separate randomization process in a split plot design

(1) For the main plot

(2) For the subplots.

In each replication main plot treatments are first randomly assign to the main plots followed by a random assignment on the ~~same~~ sub plot treatments within each main plot treatment. This procedure is applied for all replications. A possible layout of a split plot experiment with four main plot treatments, three sub plot treatments and two replications is given below

Replicate - I

b_2	b_1	b_2	b_1
b_1	b_3	b_3	b_2
b_3	b_2	b_1	b_3
a_1	a_4	a_3	a_2

Replicate - II

b_1	b_3	b_2	b_3
b_2	b_1	b_3	b_2
b_3	b_2	b_1	b_1
a_2	a_4	a_1	a_3

The above layout has the following important features :→

(1) The size of the main plot is q times the size of the sub plot.

(2) Each main plot treatment is tested r times whereas each sub plot treatment is tested pr times, thus number of times a subplot treatment

as tested will always be larger than that for the main plot. Therefore, the sub plot treatment have the more precision as compare to the main plot treatment.

Analysis :- let p levels of factor A be randomized by using the layout of RBD (or) LSD and the q levels of factor B are then randomized within each whole plot of factor A by dividing each whole plot into q subplots.

Thus, for split plot design we use the following model

$$y_{ijk} = \mu + b_i + \tau_j + e_{ij} + \gamma_k + \delta_{jk} + e'_{ijk}$$

$$i = 1, 2, \ldots r, \; j = 1, 2, \ldots p, \; k = 1, 2, \ldots q$$

Here,

τ_j, γ_k and δ_{jk} are the fixed effects due to the jth level of factor A, kth level of factor B and interaction between them respectively.

Thus, obviously

$$\sum_j \tau_j = \sum_k \gamma_k = \sum_{\substack{j \\ \forall k}} \delta_{jk} = \sum_{\substack{k \\ \forall j}} \delta_{jk} = 0$$

b_i, e_{ij}, e'_{ijk} are random components which are independently normally distributed with mean zero and respective variance such that

$$b_i \overset{iid}{\sim} N(0, \sigma_b^2)$$

$$e_{ij} \overset{iid}{\sim} N(0, \sigma_e^2)$$

$$e'_{ijk} \overset{iid}{\sim} N(0, \sigma_{e'}^2)$$

In this design the analysis can be done into two stages, at the 1st stage we use the analysis of

RBD with p treatment in r blocks. The whole plot analysis will be as follows

S.V.	d.f.	Sum of Squares
Blocks	$r-1$	$pq\sum_i(\bar{y}_{i\circ\circ}-\bar{y}_{\circ\circ\circ})^2 = SSB$
whole plot treatment (A)	$(p-1)$	$rq\sum_j(\bar{y}_{\circ j\circ}-\bar{y}_{\circ\circ\circ})^2 = SSA$
whole plot error E_I	$(r-1)(p-1)$	$q\sum_i\sum_j(\bar{y}_{ij\circ}-\bar{y}_{i\circ\circ}-\bar{y}_{\circ j\circ}+\bar{y}_{\circ\circ\circ})^2 = SSE_I$
Total	$rp-1$	$q\sum_i\sum_j(\bar{y}_{ij\circ}-\bar{y}_{\circ\circ\circ})^2$

Here, it can be shown that.

$E(MSSA) = \sigma_{e'}^2 + q\sigma_e^2 + \phi_1(\tau_j')$

$E(MSSE_I) = \sigma_{e'}^2 + q\sigma_e^2$

Under H_{01}: All the effects due to the jth level of A are same. i.e.

$H_{01}: \tau_j' = 0 \ \forall j$

i.e. is

$\phi_1 = 0$

otherwise, $\phi_1 > 0$

Thus, a test for testing H_{01} is provided by

$$F = \frac{MSSA}{MSSE_I} \sim F^{(\alpha)}_{\{(p-1),(r-1)(p-1)\}}$$

The second stage will be the analysis of the sub plot analysis within whole blocks which is as follows :-

S.V.	d.f.	Sum of Squares
Sub plot treatment (B)	$(q-1)$	$rp\sum_{k}(\bar{y}_{ook}-\bar{y}_{ooo})^2 = SSB$
Interaction (AB)	$(p-1)(q-1)$	$r\sum_{j}\sum_{k}(\bar{y}_{ojk}-\bar{y}_{ojo}-\bar{y}_{ook}+\bar{y}_{ooo})^2 = SS(AB)$
Sub Plot Error (E_{II})	$p(q-1)(r-1)$	$\sum_{i}\sum_{j}\sum_{k}(y_{ijk}-\bar{y}_{ojk}-\bar{y}_{ijo}+\bar{y}_{ojo})^2 = SS\,E_{II}$
Total between Sub Plots within whole Plots	~~(rpq-1)~~ $rp(q-1)$	$\sum_{i}\sum_{j}\sum_{k}(y_{ijk}-\bar{y}_{ijo})^2$

Complete analysis of variance table for split plot design

S.V.	d.f.	SS	MSS	E[MSSE]	F
Blocks	$(r-1)$	SS(Blocks)	MSS(Blocks)	~~$\sigma_e^2+q\sigma_e^2+\phi_I(\tau_j'^s)$~~	$MSSA = \frac{MSSA}{MSSE_I}$
Treat(A)	$(p-1)$	SSA	MSSA	$\sigma_e^2+q\sigma_e^2+\phi_I(\tau_j's)$	
Error I	$(r-1)(p-1)$	SSE_I	$MSSE_I$	$\sigma_{e'}^2+q\sigma_e^2$	
Treat (B)	$(q-1)$	SSB	MSSB	$\sigma_{e'}^2+\phi_{II}(\psi_k's)$	$\frac{MSSB}{MSSE_{II}}$
Interaction (AB)	$(p-1)(q-1)$	SS(AB)	MSS(AB)	$\sigma_e^2+\phi_{III}(\delta_{jk}'s)$	$\frac{MSS(AB)}{MSSE_{II}}$
Error II	$p(q-1)(r-1)$	SSE_{II}	$MSSE_{II}$	σ_e^2	
Total	$rpq-1$	TSS			

Merits and Demerits of Split Plot Design :-

Merits :-

(1) In this design there are two errors type I and type II where $E_I > E_{II}$ Therefore, the main effect B and interaction AB will be estimated and tested

(2) In this design we can introduce the second factor B which requires small experimental material along A at a little extra cost.

(3) If we have a choice of allocation and the factor A and B then we shall applied most important factors to the sub plot rather than whole plots.

Demerits :-

The presence of the two errors make the analysis difficult. Also, sometimes the error I may be too large.

Strip Plot Design :-

In split plot design one factor requires smaller unit as compare to the other and we can increase the precision on the factor B and factor AB as such sacrifices some precision on A. In that design the factor A is of less important than factor B. However sometimes we may have factors A and B each requiring large units.

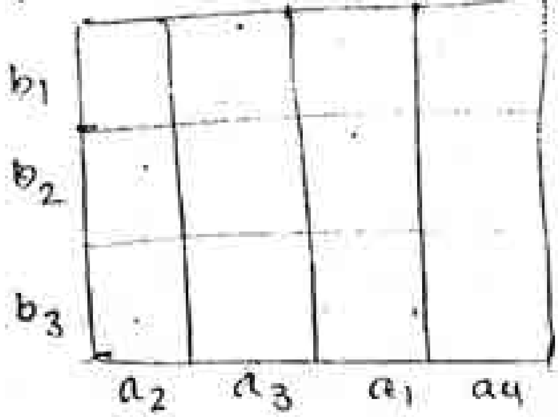
For Example :- Suppose four levels of spacing and three levels of method of ploughing are to be tested the same experiment. Therefore to accomodate both the factors in the large units we use strip plot design.

Randomization and layout :-

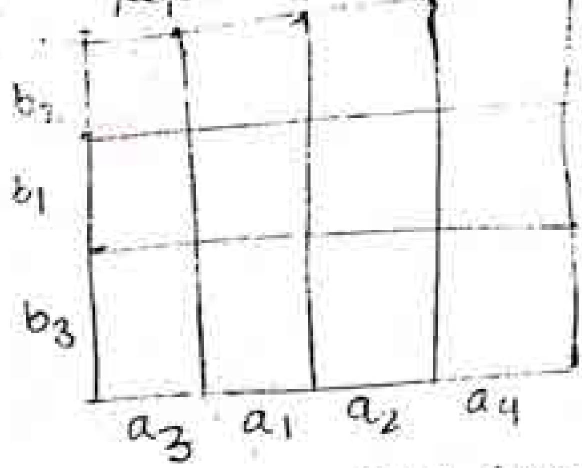
In this design each block (replicate) is divided into number of vertical and horizontal strips depending on the levels of respective factors. Let factor A represent the vertical factor with p levels, factor B represent the horizontal factor with q levels and r represent the number of replication. To layout the experiment, the experimental area is divided into r blocks. Each block is divided into q horizontal strips and q treatments are randomly assign to these strips in each of the r blocks seperately and independently. Then, each block is divided into p vertical strips ~~and p vertical strips~~ and p treatments are randomly assign to these strips in each of the r replicates seperately and independently.

A possible layout of a strip plot design with $p=4, (a_1, a_2, a_3, a_4), q=3, (b_1, b_2, b_3)$ and $r=2$.

Replicate I



Replicate II



~~Analysis~~ :→ The strip plot design sacrifies precision on the main effects of both the factors in order to provide higher precision on the interaction which will usually be more accurately determine than split plot design.

Analysis :→ In this design we have three errors for different effects. The analysis of variance for a strip plot design will be based on the model

$$y_{ijk} = \mu + r_i + \alpha_j + (r\alpha)_{ij} + \beta_k + (r\beta)_{jk} + (\alpha\beta)_{jk} + (r\alpha\beta)_{ijk}$$

where y_{ijk} denote the response or output of the yield receiving jth level of factor A, kth level of factor B in the ith replicate. Thus, $i=1,2,\ldots r$; $j=1,2,\ldots p$; $k=1,2,\ldots q$.

$(r\alpha)_{ij}$, $(r\beta)_{jk}$ and $(r\alpha\beta)_{ijk}$ are three errors which are such that

$$(r\alpha)_{ij} \overset{i.i.d.}{\sim} N(0, \sigma_1^2)$$

$$(r\beta)_{jk} \overset{i.i.d.}{\sim} N(0, \sigma_2^2)$$

$(\gamma\alpha\beta)_{ijk} \overset{iid}{\sim} N(0, \sigma_3^2)$

$\alpha_j, \beta_k, (\alpha\beta)_{jk}$ are fixed effects.

After doing some mathematical calculations we obtain the following analysis of variance table.

S.V.	d.f.	SS	MSS	E(MSS)	F
Replicate	$(r-1)$	SSR	MSSR		
Treat (A)	$(p-1)$	SSA	MSSA	$\sigma_3^2 + q\sigma_1^2 + \frac{rq\sum_j \alpha_j^2}{(p-1)}$	$\frac{MSSA}{MSSE_I}$
Error E_I	$(r-1)(p-1)$	SSE_I	$MSSE_I$	$\sigma_3^2 + q\sigma_1^2$	
Treat (B)	$(q-1)$	SSB	MSSB	$\sigma_3^2 + p\sigma_2^2 + \frac{rp\sum_k \beta_k^2}{(q-1)}$	$\frac{MSSB}{MSSE_{II}}$
Error E_{II}	$(r-1)(q-1)$	SSE_{II}	$MSSE_{II}$	$\sigma_3^2 + p\sigma_2^2$	
Treat (AB)	$(p-1)(q-1)$	SS(AB)	MSS(AB)	$\sigma_3^2 + \frac{r\sum_j\sum_k (\alpha\beta)_{jk}^2}{(p-1)(q-1)}$	$\frac{MSS(AB)}{MSSE_{III}}$
Error E_{III}	$(r-1)(p-1)(q-1)$	SSE_{III}	$MSSE_{III}$	σ_3^2	
Total	$rpq-1$	TSS			

Balanced Incomplete Block Design (BIBD)

In a confounded design we sacrifice some or total information on any of the treatment combinations to maintain the homogeneity within block. Sometimes we come across the situation where all the treatments have at equal importance then we can not afford to sacrifis information on any of them by confounding. In such situations we use BIBD. The BIBD was first purposed by Yate's in 1936.

Here incomplete block means a block which do not contain complete set of treatments. The use of incomplete block becomes necessary because as the number of treatments increases the block size increases which results and increase in the heterogeneity. when the number of replications of all pairs of treatments in an incomplete blocks is same then an important series of design known as balanced incomplete block design is obtained. In this design all the treatment effects are estimated with equal precision.

$\rightarrow \lambda = 3$

Incomplete Block Designs :→

A block design which consist v treatments in b blocks each of size k such that each of the treatment is replicated r times and each pair of treatments occurs once and only once in the same block is called incomplete block designs.

Example :→ Suppose. $v = b = 5$

$$r = k = 4$$

	I	1	2	3	4
Blocks	II	1	2	3	5
	III	1	2	4	5
	IV	1	3	4	5
	V	2	3	4	5

Parameters of BIBD :→ BIBD has five parameters which are given as

v = no. of treatments

b = no. of blocks

k = block size

r = no. of replicates for each treatment

λ = no. of ~~blocks~~ replication of each treatment pair or no. of blocks in which any pair of treatments occurs together.

Definition of BIBD :→ A BIBD is defined as, "An arrangent of v treatments in b blocks of each of size k ($k < b$) such that each treatment is replicated r times. No treatments occurs more than one in a block and each pair of treatment occurs together in λ blocks

Incidence Matrix :→ A matrix $N = (n_{ij})$ $i = 1, 2, \ldots v$, $j = 1, 2, \ldots b$ associated with any design D, where n_{ij} denotes the no. of times the ith treatment occurs in the jth block, is called incidence matrix. Thus, by the definition of the BIBD

$$n_{ij} = \begin{cases} 1 & \text{if } i\text{th treatment occurs in the } j\text{th block} \\ 0 & \text{, otherwise} \end{cases}$$

For the existence of a BIBD for the parametric relation of BIBD :-

(1) $rv = bk$

(2) $\lambda(v-1) = r(k-1)$

(3) $b \geq v$.

(iv) $b > r > \lambda$

(V) $r \geq k$ and $v > k$

(vi) $b \geq v + r - k$

(1) $rv = bk$

Proof :- Here, v is the no. of treatment in a replication therefore, rv will be total no. of plots in r replicates.

b is the no. of blocks in r replicates and k is the no. of plots in a block therefore. bk is the total no. of plots in r replicate.

Hence, $rv = bk$. Proved

(2) $\lambda(v-1) = r(k-1)$

Proof :→ Since v-treatments give rise to $\binom{v}{2}$ pairs and since each pair occurs λ times, So, total no. of times of the pairs occurs in the design is $\lambda\binom{v}{2}$.

Further, since the size of each block is k. So, each block give rise to $\binom{k}{2}$ pairs and total no. of treatment pairs in all the b blocks is $b\binom{k}{2}$

Therefore, $\lambda\binom{v}{2} = b\binom{k}{2}$

$$\lambda \cdot \frac{v!}{2!\,(v-2)!} = b\,\frac{k!}{2!\,(k-2)!}$$

$$\Rightarrow \lambda\, v(v-1) = b\,k(k-1) \quad ; \text{since } rv = bk$$

$$\Rightarrow \lambda\, \cancel{v}(v-1) = r\cancel{v}(k-1)$$

$$\Rightarrow \lambda(v-1) = r(k-1)$$

Proved

(3) $b \geq v$ (Fisher inequality)

Proof :→ Let us consider a incidence matrix

$$N = (n_{ij})_{v\times b}; \quad i = 1,2,\ldots v,\; j = 1,2,\ldots b$$

where,

$$n_{ij} = \begin{cases} 1 & ; \text{ if } i\text{th treatment occurs in the } j\text{th block} \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big\{ \text{Let } NN' = \begin{pmatrix} n_{11} & n_{12} & \cdots & n_{1b} \\ \vdots & & & \\ n_{v1} & n_{v2} & \cdots & n_{vb} \end{pmatrix} \begin{pmatrix} n_{11} & \cdots & n_{v1} \\ n_{12} & & \vdots \\ \vdots & & \vdots \\ n_{1b} & & n_{vb} \end{pmatrix} \; n_{11}^2 + n_{12}^2 + \cdots + n_{11}n_{21} + n_{12}n_{22} \cdots$$

Let
$$NN' = \begin{bmatrix} \sum_j n_{1j}^2 & \sum_j n_{1j} n_{2j} & - & - & \sum_j n_{1j} n_{vj} \\ \sum_j n_{2j} n_{1j} & \sum_j n_{2j}^2 & - & - & \sum_j n_{2j} n_{vj} \\ \vdots \\ \sum_j n_{vj} n_{1j} & \sum_j n_{vj} n_{2j} & - & - & \sum_j n_{vj}^2 \end{bmatrix}_{v \times b}$$

where, for BIBD $\sum_{j=1}^{r} n_{ij} = n_{i1} + n_{i2} + - + n_{ib} \; \forall i$
$$= r = \sum_{j=1}^{b} n_{ij}^2$$

Similarly.
$$\sum_{i=1}^{v} n_{ij} = n_{1j} + n_{2j} + - - + n_{vj}$$
$$= k = \sum_{i=1}^{v} n_{ij}^2 \; \forall j$$

Remark →
$$\begin{array}{cc} \begin{bmatrix} n_{11} & n_{12} & - - - & n_{1b} \\ n_{21} & n_{22} & \text{———} & n_{2b} \\ \vdots & & - & \\ n_{v1} & n_{v2} & \text{———} & n_{vb} \end{bmatrix}_{v \times b} & \begin{array}{l} \to r \\ \to r \\ \\ \\ \end{array} \\ \begin{array}{cc} \downarrow & \downarrow \\ k & k \end{array} & \end{array}$$

Sum of the ith row of a incedence matrix $= r$

Sum of the jth column of a incidence matrix $= k$.

$$\sum_{j=1}^{m} n_{ij} \cdot n_{lj} = \lambda (i \neq l)$$

Thus
$$NN' = \begin{bmatrix} r & \lambda & - & - & - & \lambda \\ \lambda & r & - & - & - & \lambda \\ \vdots \\ \lambda & \lambda & - & \text{———} & & r \end{bmatrix}_{v \times v}$$

$$NN' = \begin{bmatrix} r+(v-1)\lambda & \lambda & - & - & \lambda \\ r+(v-1)\lambda & r & - & & \lambda \\ \vdots & & & & \\ r+(v-1)\lambda & \lambda & - & & r \end{bmatrix}$$ $C_1 \to C_1 + C_2 \cdots C_v$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$
$R_i \to R_i - R_1 \; (i = 1, \dots)$

$$= \begin{bmatrix} r+(v-1)\lambda & \lambda & - & - & \lambda \\ 0 & (r-\lambda) & - & - & 0 \\ \vdots & & & & \\ 0 & - & 0 & - & (r-\lambda) \end{bmatrix}$$

$$= [r+(v-1)\lambda] \begin{bmatrix} 1 & \lambda & - & - & \lambda \\ 0 & (r-\lambda) & - & - & 0 \\ \vdots & & \ddots & & \\ 0 & - & 0 & - & (r-\lambda) \end{bmatrix}$$

$$\Rightarrow [r+(v-1)\lambda] \begin{bmatrix} 1 & - & - & 0 \\ \vdots & (r-\lambda) & - & - \\ \vdots & & & \\ 0 & - & - & (r-\lambda) \end{bmatrix}$$

$|NN'| = [r+(v-1)\lambda](r-\lambda)^{v-1} \neq 0$

Therefore, Rank of $NN' = v$ (Because the order of $NN' = v$)

$\Rightarrow \rho(NN') \leq \rho(N) = \min(b, v)$

But we know that

$N \geq$ Rank of NN'

$\Rightarrow b \geq v$ Proved

(iv) $b > r > \lambda$

Proof :- Since r is the no. of times ith treatment is replicated and λ is the no. of times, ith treatment occurs with jth treatment and since due to incomplete blocks the ith treatment cannot occur r times with jth treatment. Therefore

$$\lambda < r \quad \text{---} (1)$$

Again, since $v > k$ (For BIBD)

$$\Rightarrow v > \frac{rv}{b} \quad \{\because bk = rv$$

$$\Rightarrow b > r \quad \text{---} (2)$$

from eq^n (1) and (2)

$$b > r > \lambda$$ Proved.

(v) $r \geq k$ and $v > k$

Proof :- we know that

$$bk = rv$$

$$\Rightarrow \frac{b}{v} = \frac{r}{k} \quad \text{---} (1).$$

But we have $b \geq v$

$$\Rightarrow \frac{b}{v} \geq 1$$

On putting in eq^n (1) we get

$$\Rightarrow \frac{r}{k} \geq 1$$

$$\Rightarrow r \geq k$$

So, $v > k$ Proved.

(vi) $b \geq v + r - k$

Proof :→ we know that
$r \geq k$ and $v > k$ (for BIBD)

Therefore, $(r-k) \geq 0$ and $(v-k) > 0$

$$(r-k)(v-k) \geq 0$$
$$rv - kv - rk + k^2 \geq 0$$
$$bk - kv - rk + k^2 \geq 0$$
$$k(b - v - r + k) \geq 0$$
$$b - v - r + k \geq 0$$
$$b \geq v + r - k.$$

Proved

Symmetrical BIBD :→

A BIBD is said to be Symmetrical if $b = v$ and $r = k$. In this case the incidence matrix N is a square matrix.

"An arrangement of v treatments in b blocks of each of size k ($k < b$) such that each treatment is replicated r times. No treatment occurs more than one in a block and each pair of treatment occurs together in λ blocks."

~~For a~~

Theorem :→ For a Symmetrical BIBD $(r-\lambda)$ will be a perfect square when v is even.

Proof :→ we have

$$|NN'| = rk(r-\lambda)^{v-1}$$
$$\Rightarrow |N||N'| = r^2(r-\lambda)^{v-1} \quad (\text{for } r = k)$$

$\Rightarrow |N|^2 = r^2(r-\lambda)^{v-1}$

$\Rightarrow |N| = \pm r(r-\lambda)^{(v-1)/2}$

Since, the determinant of a incidence matrix N is a integer.

Therefore, $r(r-\lambda)^{(v-1)/2}$ is a integer. Also, r is the integer. Therefore $(r-\lambda)^{(v-1)/2}$ is integer. Thus, $(r-\lambda)^{(v-1)/2}$ will be a perfect square. when v is ~~given~~ even. Proved

Theorem-2 :- In a Symmetrical BIBD the no. of treatment common between any two block is λ.

OR

In a Symmetrical design every block different from the first block has exactly λ treatments for with the first block.

Proof :- let a_i be the no. of treatments common to the first and ith ~~treat~~ block and $i=1,2,-b$

Since, each of the k treatment of the 1^{st} block occurs $(r-1)$ times in the other block. Then.

$$\sum_{i=2}^{b} a_i = k(r-1) \quad (1)$$

Again, Since each of the $\binom{k}{2}$ pairs of treatments of the first block occurs $(\lambda-1)$ times in the remaining blocks.

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Hence, $\sum_{i=2}^{b} \binom{a_i}{2}$ pairs $= (\lambda - 1)\binom{k}{2}$

$\Rightarrow \sum_{i=2}^{b} a_i(a_i - 1) = (\lambda - 1)k(k-1)$.

$\Rightarrow \sum_{i=2}^{b} a_i^2 = (\lambda-1)k(k-1) + \sum_{i=2}^{b} a_i$ ——— (2)

from eq^n (1) (Σa_i) is

$\sum_{i=2}^{b} a_i^2 = (\lambda-1)k(k-1) + k(r-1)$ ——— (3)

let $\sum_{i=2}^{b} (a_i - \lambda)^2 = \sum_{i=2}^{b} (a_i^2 + \lambda^2 - 2a_i\lambda)$

$= \sum_{i=2}^{b} a_i^2 + \sum_{i=2}^{b} \lambda^2 - 2\sum_{i=2}^{b} a_i\lambda$

$= (\lambda-1)k(k-1) + k(r-1) + \underset{(b-1)\lambda^2}{\cancel{b\lambda^2}} - 2k(r-1)\lambda$

If, $r = k$, $b = v$ then

$= (\lambda-1)k(k-1) + k(k-1) + \underset{(v-1)}{\cancel{v}}\lambda^2 - 2k(k-1)\lambda$

$= k(k-1)[\lambda - \cancel{1} + \cancel{1} - 2\lambda] + (v-1)\lambda^2$

$= k(k-1)(-\lambda) + (v-1)\lambda^2$

$= -\lambda k(k-1) + \lambda^2(v-1)$

If $\lambda(v-1) = r(k-1)$ then and $r = k$

$= -\lambda k(k-1) + \lambda . k(k-1)$

$= 0$

$\Rightarrow \sum_{i=2}^{b} (a_i - \lambda)^2 = 0$

$\Rightarrow \boxed{a_i = \lambda}$

Proved.

1.

Resolvable B.I.B.D :→

A BIBD with parameters v, b, r, λ, k is said to be resolvable BIBD if b blocks can be divided into r groups for the sets of $\frac{b}{r}$ blocks each. $\frac{b}{r}$ is an integer such that $\frac{b}{r}$ blocks forming any of the sets gives a complete replication of all the v treatments.

Example :→ let us consider a BIBD with parameter $v=4, b=6, r=3, k=2, \lambda=1$

1	1	2	Ist group
2	3	4	
3	1	3	IInd group
4	2	4	
5	1	4	IIIrd group
6	2	3	

V. Imp

Theorem :→ Bose Inequality

For a resolvable BIBD prove that
$$b \geq v+r-1$$

Proof :→ Suppose, $n = \frac{b}{r}$

$$\Rightarrow b = nr$$

we know that

$bk = vr$

Therefore,

$n r k = r v$

$\Rightarrow v = nk$ ——— (1)

Again, consider

$\lambda(v-1) = r(k-1)$

$\Rightarrow \lambda(nk-1) = r(k-1)$ {from eq^n (1)

$\Rightarrow r = \frac{\lambda(nk-1)}{k-1}$

$\Rightarrow = \lambda n + \frac{\lambda(n-1)}{(k-1)}$

$\Rightarrow r - \lambda n = \frac{\lambda(n-1)}{(k-1)}$ ——— (2)

Since, r is an integer, λ and n are also integer, Therefore, from eq^n (2) we conclude that $\frac{\lambda(n-1)}{(k-1)}$ must be an integer.

Let us consider

$b < v + r - 1$ ——— (3)

$\Rightarrow b - r < v - 1$

$\Rightarrow nr - r < v - 1$ $\because b = n\lambda$

$\Rightarrow (n-1)r < v - 1$

$\Rightarrow \lambda r(n-1) < \lambda(v-1)$

$\Rightarrow \lambda r(n-1) < r(k-1)$ {$\because \lambda(v-1) = r(k-1)$}

$\Rightarrow \lambda(n-1) < (k-1)$

$\Rightarrow \frac{\lambda(n-1)}{(k-1)} < 1$

which is a contradiction of the fact that $\frac{\lambda(n-1)}{(k-1)}$ is an integer.

Hence, our assumption is wrong and we must have,

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$b \geq v + r - 1$

Proved

Statistical Analysis for (BIBD) (Intra Block Anaylysis (recovery) :→

This analysis is known as intra block analysis b/o because it is carried out without recovery of intrablock information. Let y_{ij} be the response of j^{th} block receiving i^{th} treatment if it occurs. Here, we consider the model

$$y_{ij} = \mu + \alpha_i + \beta_j + e_{ij} \text{ ——— (1)} \quad \begin{array}{l} i = 1, 2, \ldots v \\ j = 1, 2, \ldots b \end{array}$$

where, μ = general mean effect

α_i = additional effect due to i^{th} treatment

β_j = additional effect due to j^{th} block

e_{ij} = intrablock error assume to be i.i.d.

i.e. $e_{ij} \sim N(0, \sigma_e^2)$

$\sum_i \alpha_i = 0, \ \sum_j \beta_j = 0$

The least square estimates of μ, α_i and β_j can be obtained by minimizing residual sum of squares

$$S = \sum_i \sum_j n_{ij} (y_{ij} - \mu - \alpha_i - \beta_j)^2 \text{ ——— (2)}$$

where, $n_{ij} = \begin{cases} 1 \ ; \text{when } i^{th} \text{ treatment occur in } j^{th} \text{ block} \\ 0 \ ; \text{otherwise} \end{cases}$

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$$\frac{\partial S}{\partial \mu} = 0 \Rightarrow 2\sum_i \sum_j n_{ij}(y_{ij} - \mu - \alpha_i - \beta_j) = 0 \quad \text{--- (3)}$$

$$\frac{\partial S}{\partial \alpha_i} = 0 \Rightarrow 2\sum_j n_{ij}(y_{ij} - \mu - \alpha_i - \beta_j) = 0 \quad \text{--- (4)}$$

$$\frac{\partial S}{\partial \beta_j} = 0 \Rightarrow 2\sum_i n_{ij}(y_{ij} - \mu - \alpha_i - \beta_j) = 0 \quad \text{--- (5)}$$

Now, from eq^n (3) we have

$$\sum_i \sum_j n_{ij} y_{ij} = \sum_i \sum_j n_{ij}\mu + \sum_i \sum_j n_{ij}\alpha_i + \sum_i \sum_j n_{ij}\beta_j$$

$$\Rightarrow \sum_i \sum_j n_{ij} y_{ij} = rv\mu + r\underbrace{\sum_i \alpha_i}_{0} + k\underbrace{\sum_j \beta_j}_{0}$$

$$\Rightarrow \hat{\mu} = \frac{\sum_i \sum_j n_{ij} y_{ij}}{rv} = \frac{G}{rv} \quad \text{--- (6)}$$

From eq^n (4) & (5) we get

$$\sum_j n_{ij} y_{ij} = r\hat{\mu} + r\hat{\alpha}_i + \sum_j n_{ij}\hat{\beta}_j \quad \text{--- (7)}$$

$$\sum_i n_{ij} y_{ij} = k\hat{\mu} + \sum_i n_{ij}\hat{\alpha}_i + k\hat{\beta}_j \quad \text{--- (8)}$$

Let

$$\sum_{i=1}^{v} n_{ij} y_{ij} = n_{1j}y_{1j} + n_{2j}y_{2j} + \cdots + n_{vj}y_{vj}$$

$= B_j$ = Total response due to the j^{th} block

Now, ~~from eq^n (7) & (8)~~

Again, let

$$\sum_{j=1}^{b} n_{ij} y_{ij} = n_{i1}y_{i1} + n_{i2}y_{i2} + \cdots + n_{ib}y_{ib}$$

$= T_i$ = Total response due to i^{th} treatment

Therefore, from eq^{n}s (7) & (8)

$$T_i = r\hat{\mu} + r\hat{\alpha_i} + \sum_j n_{ij}\hat{\beta_j} \qquad \text{——— (9)}$$

$$B_j = k\hat{\mu} + \sum_i n_{ij}\hat{\alpha_i} + k\hat{\beta_j} \qquad \text{——— (10)}$$

Let us consider a quantity

$$Q_i = \left(T_i - \sum_j \frac{n_{ij} B_j}{k}\right)$$

On putting the values of T_i and B_j

$$= \cancel{r\hat{\mu}} + r\hat{\alpha_i} + \cancel{\sum_j n_{ij}\hat{\beta_j}} - \cancel{\sum_j n_{ij}\hat{\mu}} - \sum_j n_{ij}\left(\frac{\sum_i n_{ij}\hat{\alpha_i}}{k}\right)$$

$$- \cancel{\sum_j n_{ij}\hat{\beta_j}}$$

$$= r\hat{\alpha_i} - \frac{1}{k}\sum_j n_{ij}[n_{1j}\hat{\alpha_1} + n_{2j}\hat{\alpha_2} + \dots + n_{ij}\hat{\alpha_i} + \dots - n_{vj}\hat{\alpha_v}]$$

$$= r\hat{\alpha_i} - \frac{1}{k}[\hat{\alpha_1}\lambda + \hat{\alpha_2}\lambda + \dots + \hat{\alpha_i}r + \dots + \hat{\alpha_v}\lambda]$$

$$= r\hat{\alpha_i} - \frac{\lambda}{k}\underbrace{\sum_{i=1}^{v}\hat{\alpha_i}}_{0} - \frac{r\hat{\alpha_i}}{k} + \frac{\lambda\hat{\alpha_i}}{k}$$

$$= r\hat{\alpha_i} - \frac{r\hat{\alpha_i}}{k} + \frac{\lambda\hat{\alpha_i}}{k}$$

$$= \frac{1}{k}[rk - r + \lambda]\hat{\alpha_i}$$

$$= \frac{1}{k}[r(k-1) + \lambda]\hat{\alpha_i}$$

$$= \frac{1}{k}[\underline{\lambda(v-1) + \lambda}]\hat{\alpha_i}$$

$$Q_i = \frac{\lambda v}{k}\hat{\alpha_i} \qquad \text{——— (11)}$$

The quantity Q_i is called adjusted treatment total.

Now, suppose we want to test

$H_0: \alpha_i = 0 \ \forall\ i = 1, 2, \dots, v$

for this purpose we first calculate sum of squares due to regression when all the parameters are taken into consideration.

Thus, $R(\hat{\mu}, \hat{\alpha}_i, \hat{\beta}_j) = \hat{\mu}G + \sum_i \hat{\alpha}_i T_i + \sum_j \hat{\beta}_j B_j$

$$= \hat{\mu}G + \sum_i \hat{\alpha}_i T_i + \sum_j \hat{\beta}_j \left[\frac{B_j}{K} - \hat{\mu} - \frac{1}{K}\sum_i n_{ij}\hat{\alpha}_i\right]$$

From eqn (10)

$$= \cancel{\hat{\mu}G} + \sum_i \hat{\alpha}_i T_i + \sum_j \frac{B_j^2}{K} - \hat{\mu}\sum_j B_j - \frac{1}{K}\sum_j B_j \sum_i n_{ij}\hat{\alpha}_i$$

$$= \sum_i \hat{\alpha}_i T_i + \sum_j \frac{B_j^2}{K} - \frac{1}{K}\sum_i \hat{\alpha}_i \sum_j n_{ij} B_j$$

$$= \sum_j \frac{B_j^2}{K} + \sum_i \hat{\alpha}_i \left[T_i - \frac{1}{K}\sum_j n_{ij} B_j\right]$$

$$= \sum_j \frac{B_j^2}{K} + \sum_i \hat{\alpha}_i Q_i \qquad \left[\hat{\alpha}_i = \frac{KQ_i}{\lambda v}\right]$$

$$= \sum_j \frac{B_j^2}{K} + \sum_i \frac{K}{\lambda v} Q_i^2 \longrightarrow (12)$$

Sum of square due to regression under null hypothesis i.e. when only two parameters are taken into consideration

$R(\mu, \beta_j) = \hat{\mu}G + \sum_j \hat{\beta}_j B_j$

$$= \hat{\mu}G + \sum_j \left[\frac{B_j}{K} - \hat{\mu}\right] B_j$$

From eqn (10) on putting $\alpha_i = 0$; $\hat{\mu}G$

$$= \sum_j \frac{B_j^2}{K} \longrightarrow (13)$$

Thus, using eqns (12) & (13) the adjusted treatment sum of squares is given by

$$R(\mu, \alpha_i, \beta_j) - R(\mu, \beta_j) = \frac{k}{\lambda v} \sum_i Q_i^2$$

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The Sum of Squares due to blocks ignoring treatments (i.e.) i.e. unadjusted block Sum of Squares is given by.

$$\sum_j \frac{B_j^{\cdot 2}}{k} - \frac{G^2}{rv} \quad \left[\frac{G^2}{rv} = \text{correction factor} \right]$$

Thus the Anova table for intra block analysis of BIBD is given by

ANOVA TABLE

Source of variation	degree of freedom	Sum of Squares
Blocks (unadjusted)	$(b-1)$	$\sum_j \frac{B_j^2}{k} - \frac{G^2}{rv}$
Treatments (adjusted)	$(v-1)$	$\frac{k}{\lambda v} \sum_i Q_i^2$
Intra block error	$bk-b-v+1$	by difference
Total	$bk-1$	$\sum_i \sum_j (n_{ij}\, y_{ij}^2) - \frac{G^2}{rv}$

Graeco - Latin Square :-

This is an another name for pair of orthogonal Latin squares superimposed on one another. Here, one square is represented by Greek letters and other by latin letters. In this arrangement Greek (or latin) letters occurs once.

in each row and once in each column, and once with each latin letter and once with each greek letter.

A $m \times m$ Græco latin square is an incomplete four-way layout with all the four factors at the same level m and observations are taken on only m^2 of the possible m^4 of treatment combination.

For Ex :→

$$\begin{bmatrix} A & B & C \\ B & C & A \\ C & A & B \end{bmatrix}_{3\times3} \cdot \begin{bmatrix} \gamma & \beta & \alpha \\ \alpha & \gamma & \beta \\ \beta & \alpha & \gamma \end{bmatrix}$$

$$\begin{bmatrix} A\gamma & B\beta & C\alpha \\ B\alpha & C\gamma & A\beta \\ C\beta & A\alpha & B\gamma \end{bmatrix}_{3\times3}$$

In Græco-latin square the anova table will have five components rows, columns, latin letters, Greek letters and error. The degrees of freedom for each of the first four components will be $(m-1)$ and ~~four~~ for error will be $(m-1)(m-3)$.

Remark :→ A field can consist of either finite or infinite number of elements. A field with infinite number of elements is called infinite field. If the number of elements i.e. marks of field is finite ~~then~~ the field is called finite or Galois field.

Finite Field and the Design of Experiment

The properties of the finite field or Galois field makes construction and analysis of several types of design very convenient. These properties are as follows :-

(1) A rule is made that any positive number is equal to the remainder R when N is divided by any ~~integer~~ number p and R is written as

$$R = N \bmod p$$

(2) If p is a prime number then all the four operations addition, subtraction, multiplication and division of the elements are possible.

(3) when any element of prime modulo is multiplied in term by all its non-zero elements each time a different product is obtained. This ensures that all possible divisions occurs. This is not true when p is not prime. All divisions are also not possible in this case. when division is possible the elements are said to form a Galois field.

(4) There is atleast one element in every field, different powers of which give the different non-zero elements of the field. Such an element is called the permitive root of the field.

Ex :- when $p = 7$
the permitive root is 3 because
$3^0 = 1$, $3^1 = 3$, $3^2 = 2$, $3^3 = 6$, $3^4 = 4$, $3^5 = 5$, $3^6 = 1$

Thus, $x^{p-1} = 1$
where, x is any element of the field.

Though the above device ensures that an infinite sets of numbers could be reduced to only p elements, they can be manipulated just like the infinite set of numbers. However, it is not true when p is not prime. But, when p is a power of prime then such treatments are again possible by defining the elements in the following modified ways.

(1) Let the number of elements be $S = p^n$ (S is power of prime) where p is a prime and n is integer.

Let us now define S elements consisting of zero and (S-1) polynomials up to (n-1) degree and they can be written by using any symbol α (let). The coefficients in the polynomials are the elements of the modulo p. With these elements as coefficients the total number of such polynomials is (S-1).

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Ex: $\rightarrow$ $S = 3^2$ then the elements of the modulo 3 has the coefficients $0, 1, 2, \alpha, \alpha+1, \alpha+2, 2\alpha, 2\alpha+1, 2\alpha+2$.

(ii) These elements can be added and subtracted, under modulo p but they cannot be multiplied and divided unless a device for reducing the polynomials is obtained. Such a device consist of choosing an irreducable polynomials of degree n or more is equated to the remainder, when it is divided by the irreducable polynomial. Such reduced polynomials are called minimum functions.

These functions are not easy to obtain. However, some of them which may be required are given below

Galoicous field GF(S)	Minimum function
2^2	$\alpha^2+\alpha+1$
2^3	$\alpha^3+\alpha+1$
3^2	$\alpha^2+\alpha+2$

There may be more than one minimum function in a field

(iii) The non-zero elements can be obtained from different powers of α up to α^{p^n-2}

and $\alpha^{p^n - 1}$ is always 1.

For E.x :→ In the ~~the~~ 3^2 field with minimum function $(\alpha^2+\alpha+2)$ is

$\alpha^0 = 1, \alpha^1 = \alpha, \alpha^2 = 2\alpha+1$

$\alpha^3 = 2\alpha+2, \alpha^4 = 2, \alpha^5 = 2\alpha, \alpha^6 = \alpha+2, \alpha^7 = \alpha+1, \alpha^8 = 1$

Multiplication and division become very easy by using such powers as elements.

Ex :→ $\alpha^3 = 2\alpha+2, \alpha^6 = \alpha+2$

$$\begin{aligned}(2\alpha+2)(\alpha+2) &= \alpha^3 \cdot \alpha^6 \\ &= \alpha^9 \\ &= \alpha^8 \cdot \alpha \\ &= 1 \cdot \alpha = \alpha\end{aligned}$$

imp

Construction of Mutually Orthogonal Latin Squares (MOLS) and BIBD :→

construction of orthogonal-series 1 (OS1)

~~cols~~ when s is a prime or power of a prime the following series of a design can be constructed GF(s)

$$v = s^2$$
$$b = (s+1)s$$
$$r = (s+1)$$
$$k = s$$
$$\lambda = 1$$

E.x :→ let us consider $s = 4 = 2^2$

$$v = s^2 = 16$$
$$b = 5 \times 4 = 20$$
$$r = 5$$
$$k = 4$$
$$\lambda = 1$$

Thus design with these parameter can be constructed as follows :

Remark :- If we are given the prime or power of prime then we get these permitive roots

$3 \to 2$ $\quad$ $17 \to 3$
$5 \to 2$ $\quad$ $19 \to 2$
$7 \to 3$ $\quad$ $23 \to 2$
$11 \to 2$
$13 \to 2$

Step-1 :- Construct complete set of MOLS of order 4. Since, $S = q = 2^2$ is a power of prime therefore we have $q-1 = 3$ latin squares in complete sets of MOLS.

First we construct three latin squares for this we write the elements of Galois field (4) as $\{0, 1, \alpha, 1+\alpha\}$.

Now, we follow the following additional modulo rule

	0	1	α	$1+\alpha$
0	0	1	α	$1+\alpha$
1	1	0	$1+\alpha$	α
α	α	$1+\alpha$	0	1
$1+\alpha$	$1+\alpha$	α	1	0

let us assume $0 \to A$, $1 \to B$, $\alpha \to C$, $1+\alpha \to D$

then we get the latin square

(1).

A B C D
B A D C
C D A B
D C B A

$\underbrace{\begin{matrix} ABCD & ABCD \\ CDAB & CDAB \\ BADC & DCBA \\ DCBA & BADC \end{matrix}}$ are the cyclic rotation

To get the complete set of ~~mols~~ MOLS we fixed the first row and give the cyclic rotation to the other rows. Therefore, the complete set of

(2)	α	β	γ	δ	(1)
	γ	δ	α	β	3
	δ	γ	β	α	4
	β	α	δ	γ	(2)

(3)	P	Q	R	S	1
	S	R	Q	P	4
	Q	P	S	R	2
	R	S	P	Q	3

<u>Step-2</u> :→ To write down the design, we write rows as blocks.

<u>Step-3</u> :→ Then columns as blocks

<u>Step-4</u> :→ By super imposing (1) on (4) we write first row with those numbers coming along-with first letter, second row with those numbers coming along with second letter and so on.

<u>Step-5</u> :→ Same as Step-4 by superimposing (2) on (4).

(4)

1	1	2	3	4		
2	5	6	7	8		Step - 2
3	9	10	11	12		
4	13	14	15	16		
5	1	5	9	13		
6	2	6	10	14		Step - 3
7	3	7	11	15		
8	4	8	12	16		
9	1	6	11	16	$\rightarrow_A$	Position of A on OSI (1)
10	2	5	12	15	B	
11	3	8	9	14	C	Step - 4
12	4	7	10	13	D	
13	1	7	12	14	α	
14	2	8	11	13	β	(2lon (4) ⑤
15	3	5	10	16	γ	
16	4	6	9	15	δ	
17	1	8	10	15	P	
18	2	7	9	16	Q	⑥
19	3	6	12	13	R	
20	4	5	11	14	S	

Sm/ Construction of Orthogonal Series 2 (OS2) :-

This is also equivalent to BIBD when s is a power of prime. The following series of design can be constructed

$$v = b = s^2 + s + 1$$
$$k = r = s + 1$$
$$\lambda = 1$$

This series can also be constructed with the help of complete set of MOLS. To construct OS2 we need of OS1 which we describe in the following example :- let us suppose

$$s = 4 = 2^2$$
$$v = b = 4^2 + 4 + 1 = 21$$
$$k = r = 4 + 1 = 5$$
$$\lambda = 1$$

Step-1 :- write down OS1.

Step-2 :- Here, number of columns $k = 5$ whereas in OS1 $k = 4$ also number of treatments in OS2 is 21 whereas in OS1 $v = 16$. Therefore, we have to add five more treatments namely 17, 18, 19, 20, 21. Now, in the new fifth column we add treatment 17 four times, treatment 18 four times, --- treatment 21 four times such that we have $k = 5$ and $v = 21$.

Step-3 :- In OS2 $b = 21$ but in OS1 $b = 20$; therefore, to make $b = 21$ we add a block of size 5 new treatments 17, 18, 19, 20, 21 such that $b = 21$ and we can check that $r = 5$ and $\lambda = 1$

	k				
1	1	2	3	4	17
2	5	6	7	8	17
3	9	10	11	12	17
4	13	14	15	16	17
5	1	5	9	13	18
6	2	6	10	14	18
7	3	7	11	15	18
8	4	8	12	16	18
9	1	6	11	16	19
10	2	5	12	15	19
11	3	8	9	14	19
12	4	7	10	13	19
13	1	7	12	14	20
14	2	8	11	13	20
15	3	5	10	16	20
16	4	6	9	15	20
17	1	8	10	15	21
18	2	7	9	16	21
19	3	6	12	13	21
20	4	5	11	14	21
21	17	18	19	20	21

Construction of BIBD through Difference

Method -: i)

let $x_0, x_1, \ldots x_{s-1}$ be the elements of the Galoious field $GF(s)$ of order s where s is the power or power of prime. If we find k elements out of s which are $x_{i_1}, x_{i_2}, \ldots x_{i_k}$ such that $k(k-1)$ differences $[(x_{i_l} - x_{i_m}), l \neq m]$ are contmoun null elements of the Galoious field each occuring exactly ~~λ~~

$$\lambda = \frac{k(k-1)}{s-1}$$

Then, we can construct a symmetrical BIBD with parameters $v = b = s$ and $r = k$ and $\lambda = \frac{k(k-1)}{s-1}$.

The blocks can be obtained by adding to the enitial block the non-null elements of $GF(s)$ in succession.

Remark -: i) Here, number of differences will be ${}^kC_2 = \frac{k(k-1)}{2}$ but we ~~can~~ take both side differences therefore, no. of differences will be $k(k-1)$.

Example -: let $k = 3, v = 7$ then

The permitive root of 7 is 3. Then, we know that first row of BIBD will be $= 3^0, 3^2, 3^4$

$= 1, 2, 4$

$=$ under modulo 7

Here, it is notable that we take always even power in increasing order (ascending order) of permitive root. Now, the differences among elements are as 1, 2, 4 of GF(7).

$1 - 2 = -1 = 6$ under mod 7 $\qquad 7-1=6$

$1 - 4 = -3 = 4$ " $\qquad 7-3=4$

$2 - 1 = 1$

$2 - 4 = -2 = 5$ " $\qquad 7-2=5$

$4 - 1 = 3$

$4 - 2 = 2$

All the elements are non-null elements of GF(7) and each occurs only once because here, $\lambda = \frac{k(k-1)}{s-1} = \frac{3\times 2}{6} = 1$

Hence, the BIBD will be

1	2	4
2	3	5
3	4	6
4	5	0
5	6	1
6	0	2
0	1	3

<u>Ques</u> :- Construct a BIBD with $v = 11$, $k = 5$ with the method of difference.

Then the primitive root of 11 is 5, Then, we know that first row of C = Cr will be $5^0, 5^2, \ldots, 5^6, 5^8$

Complementary BIBD :→

A complementary BIBD can be obtained by a given BIBD by writting those treatments which have not appeared with in a block. If the parameters of original BIBD are v, b, k, λ, r then the parameters of the complementary BIBD will be

$v' = v$, $b' = b$, $r' = b - r$, $k' = v - k$, $\lambda' = b - 2r + \lambda$

Ex :→ $k = 3$, $v = 7$. initial block

1	2	4	3	5	6	0
2	3	5	1	4	6	0
3	4	6	1	2	5	0
4	5	0	1	2	3	6
5	6	1	2	3	4	0
6	0	2	1	3	4	5
0	1	3	2	4	5	6

Residual BIBD :→

Given a symmetrical BIBD we get another BIBD called residual BIBD by deleting the first block and deleting the treatments belonging to the initial block from the remaining $(b-1)$ blocks. The parameters of the residual BIBD will be $v' = v - k$, $b' = b - 1$, $r' = r$, $k' = k - \lambda$

$\lambda' = \lambda$

3	5
3	6
5	0

5 6

6 0

6 3

Derived BIBD → from a symmetrical BIBD we can get another BIBD called derived BIBD by omitting (cutting) the initial block and retaining only (take) those treatments in the remaining $(b-1)$ block which have appeared in the initial block.

2

4

4 2

1

2

1

The parameters of the derived BIBD will be
$v' = k,\ b' = b-1,\ r' = r-1,\ k' = \lambda,\ \lambda' = \lambda - 1$

Remark →

The complementary BIBD exist for all BIBD but residual and derived BIBD exist only for symmetrical BIBD.

Association Scheme. → Given v treatments, a relation satisfying the following conditions is said to be an association scheme with m groups.

(1) Any two treatments α and β are either I, II, - - - or m^{th} associates and their relationship is symmetrical. we denote $(\alpha, \beta) = i$ when α and β are i^{th} associate.

(2) Each treatment α has n_i, i^{th} associates, the number n_i being independent of α.

(3) If $(\alpha, \beta) = i$, the number of treatments γ that satisfy simultaneously $(\alpha, \gamma) = j$ and $(\beta, \gamma) = j'$ is $p^i_{jj'}$, and this number is independent of α and β. Further,

$$p^i_{jj'} = p^i_{j'j} .$$

~~The numbers~~

The number v, n_i, $p^i_{jj'}$ are called parameters of the association scheme

Partially Balanced Incomplete Block Design (PBIBD) :→

A PBIBD with m associate groups is an arrangement of v treatments in b blocks of size $K (< v)$ such that

(i) Every treatment occurs atmost once in a set.

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(2) Every treatment is replicated r times.

(3) Two treatments α and β replicates λ_i times if $(\alpha, \beta) = i$ and λ_i is independent of treatments α and β.

v, b, r, k, λ_i are called parameters of P.B.I.B.D.

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