C

Ex→: Let $X_1, X_2 \cdots X_n$ be a random sample from $N(\mu, \sigma^2)$, σ^2 being known, to test $H_0: \mu = \mu_0$ against

(i) $H_1: \mu \neq \mu_0$

(ii) $H_1: \mu > \mu_0$

(iii) $H_1: \mu < \mu_0$

develop L-R Test of size 'α'.

Soln→: (i) $H_1: \mu \neq \mu_0$

Here, $\Theta_0 = \{\mu_0\}$, $\Theta_1 = \{\mu \mid -\infty < \mu < \infty\}_{\mu \neq \mu_0}$

$\Theta = \{\mu \mid -\infty < \mu < \infty\}$

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$$\sup_{\mu \in \Theta_0} L(\underset{\sim}{x}; \mu) = L(\underset{\sim}{x}, \mu_0) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu_0)^2}$$

and

$$\sup_{\mu \in \Theta} L(\underset{\sim}{x}, \mu) = L(\underset{\sim}{x}, \bar{x})$$ Since m.l.e of μ for $\mu \in \Theta$ is $\bar{x}$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

Thus

$$\lambda = \frac{\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu_0)^2}}{\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2}}$$

$$= e^{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i-\mu_0)^2 - \sum_{i=1}^{n}(x_i-\bar{x})^2\right]}$$

$$= e^{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i^2+\mu_0^2-2\mu_0 x_i) - \sum_{i=1}^{n}(x_i^2+\bar{x}^2-2x_i\bar{x})\right]}$$

$$= e^{-\frac{1}{2\sigma^2}\left[\cancel{\sum x_i^2} + n\mu_0^2 - 2\mu_0 n\bar{x} - \cancel{\sum_{i=1}^{n} x_i^2} - n\bar{x}^2 + 2\bar{x}n\bar{x}\right]}$$

$$= e^{-\frac{1}{2\sigma^2}\left[n\mu_0^2 - 2\mu_0 n x + n x^2\right]}$$

$$= e^{-\frac{n}{2\sigma^2}\left(\mu_0^2 - 2\mu_0\bar{x} + \bar{x}^2\right)}$$

$$= e^{-\frac{n}{2\sigma^2}(\bar{x}-\mu_0)^2}$$

Now,

$$\lambda \leq \lambda_0$$

$$\Rightarrow \exp\left[\frac{-n}{2\sigma^2}(\bar{x}-\mu_0)^2\right] \le \lambda_0$$

$$\Rightarrow \frac{-n}{2\sigma^2}(\bar{x}-\mu_0)^2 \le \log\lambda_0$$

$$\Rightarrow \frac{n}{\sigma^2}(\bar{x}-\mu_0)^2 \ge -2\log\lambda_0$$

$$\Rightarrow \left(\frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma}\right)^2 \ge -2\log\lambda_0$$

$$\Rightarrow \left|\frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma}\right| \ge (-2\log\lambda_0)^{\frac{1}{2}} = \lambda_1 \text{ (say)}$$

The critical region of L-R test is

$$\omega = \left\{ \underset{\sim}{x} : \left|\frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma}\right| \ge \lambda_1 \right\}$$

where λ_1 is such that

$$P_{\mu_0}(\omega) = \alpha$$

$H_0 : \mu = \mu_0$

Now under H_0 such that $\mu = \mu_0$ the statistic under $\frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma}$ is normal $(0,1)$ random variable. Therefore $\lambda_1 = z_{\alpha/2}$ where $z_{\alpha/2}$ such that

$$\int_{z_{\alpha/2}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz = \alpha/2.$$

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(ii) $H_1: \mu > \mu_0$

Here,

$$\Theta_0 = \{\mu_0\}, \quad \Theta_1 = \{\mu: \mu_0 < \mu < \infty\}$$

$$\Theta = \{\mu: 0 \le \mu \le \infty\}$$

Now

$$\sup_{\mu \in \Theta_0} L(\underset{\sim}{x}, \mu) = L(\underset{\sim}{x}, \mu_0) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_i (x_i - \mu_0)^2}$$

Note that the m.l.e of all $\mu \in \Theta$, is

$$\hat{\mu} = \begin{cases} \mu_0 & \text{if } \bar{x} < \mu_0 \\ \bar{x} & \text{if } \bar{x} \ge \mu_0 \end{cases}$$

$$\therefore \sup_{\mu \in \Theta} L(\underset{\sim}{x}, \mu) = \begin{cases} \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_i (x_i - \mu_0)^2} & \text{if } \bar{x} < \mu_0 \\ \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_i (x_i - \bar{x})^2} & \text{if } \bar{x} > \mu_0 \end{cases}$$

Hence

$$\lambda = \begin{cases} 1 & \text{if } \bar{x} < \mu_0 \\ e^{-\frac{n}{2\sigma^2}(\bar{x} - \mu_0)^2} & \text{if } \bar{x} \ge \mu_0 \end{cases}$$

Since, $0 \le \lambda \le 1$, λ_0 must be necessarily be less than 1 it means that the observations $\underset{\sim}{x}$ for which

$\bar{x} < \mu_0$ must belong to the acceptance region. As regards to those $\underset{\sim}{x}$ for which $\bar{x} \geq \mu_0$ the condition 'λ' less than equal to is equalient to $\frac{\sqrt{n}.(\bar{x}-\mu_0)}{\sigma} \geq \lambda_1$

Thus the C-R of L-R-test is given by

$$\omega = \left\{ \underset{\sim}{x} : \frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma} \geq \lambda_1 \right\}$$

where, λ_1 is such that $P_{\mu_0}(\omega) = \alpha$

Since under H_0 such that $\mu = \mu_0$ $\{H_0 : \mu = \mu_0$

$\therefore$ $\frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma}$ is standard normal variate

So $\lambda_1 = z_\alpha$ where z_α is s.t

$$\int_{z_\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = \alpha.$$

(iii) $H_1 : \mu < \mu_0$

here

$$\Theta_0 = \{\mu_0\}, \quad \Theta_1 = \{\mu : -\infty < \mu < \mu_0\}$$

$$\Theta = \{\mu : -\infty < \mu \leq 0\}$$

Now

$\sup\limits_{\mu \in \Theta_0} L(\underset{\sim}{x}, \mu) = L(\underset{\sim}{x}, \mu_0) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_i (x_i-\mu_0)^2}$

Note that, the M.L.E of all $\mu \in \Theta_1$ is

$$\therefore \hat{\mu} = \begin{cases} \mu_0 & \text{if } \bar{x} > \mu_0 \\ \bar{x} & \text{if } \bar{x} \le \mu_0 \end{cases}$$

$$\therefore \sup_{\mu \in \Theta} L(\underset{\sim}{x}, \mu) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_i (x_i - \mu_0)^2} \text{ if } \bar{x} > \mu_0$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_i (x_i - \bar{x})^2} \text{ if } \bar{x} \le \mu_0$$

Hence $\lambda = \begin{cases} 1 & \text{if } \bar{x} > \mu_0 \\ e^{-\frac{n}{2\sigma^2}(\bar{x} - \mu_0)^2} & \text{if } \bar{x} \le \mu_0 \end{cases}$

Since $0 \le \lambda \le 1$, λ_0 must be necessarily be less than 1 it means that the observations $\underset{\sim}{x}$ for which $\bar{x} \geq \mu_0$ must belong to the acceptance region. As regards to those $\underset{\sim}{x}$ for which $\bar{x} \le \mu_0$ the condition λ less than equal to λ_0 is equalent to $\frac{\sqrt{n}(\bar{x} - \mu_0)}{\sigma} \le \lambda_1$

Thus the C-R of L-R test is given by

$$\omega = \left\{ \underset{\sim}{x} : \frac{\sqrt{n}(\bar{x} - \mu_0)}{\sigma} \le \lambda_1 \right\}$$

where λ_1 s.t $P_{\mu_0}(\omega) = \alpha$ Since under H_0 such that $\mu = \mu_0$ $\frac{\sqrt{n}(\bar{x} - \mu_0)}{\sigma}$ is s.n.v so $\lambda_1 = z_\alpha$ where z_α s.t

$$\int_{-\infty}^{z_\alpha} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = 1 - \alpha$$

Note:-

It is to be noted that all the above L-R test are same as the UMP test.

Q:- Let $X_1, X_2 \ldots X_n$ be a random sample from $N(\mu, \sigma^2)$, μ & σ^2 both being unknown developed L-R test of size 'α' for testing $H_0: \mu \neq \mu_0$ against H_1:

(i) $H_1: \mu \neq \mu_0$

(ii) $H_1: \mu > \mu_0$ $\quad\left\{\sigma^2 = s^2\right\}$

(iii) $H_1: \mu < \mu_0$

Soln:- $H_1: \mu \neq \mu_0$

Here $\Theta_0 = \{\mu_0\}$, $\Theta_1 = \{\mu \mid -\infty < \mu < \infty\}, \mu \neq \mu_0$

$$\Theta = \{\mu \mid -\infty < \mu < \infty\}$$

$$\sup_{\mu \in \Theta_0} L(\underset{\sim}{x}, \mu) = L(\underset{\sim}{x}, \mu_0) = \left(\frac{1}{s\sqrt{2\pi}}\right)^n e^{-\frac{1}{2s^2}\sum_{i=1}^{n}(x_i-\mu_0)^2}$$

and

$\sup_{\mu \in \Theta} L(\underset{\sim}{x}, \mu) = L(\underset{\sim}{x}, \bar{x})$ Since M.L.E of μ for $\mu \in \Theta$ is $\bar{x}$

$$= \left(\frac{1}{s\sqrt{2\pi}}\right)^n e^{-\frac{1}{2s^2}\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

Thus

$$\lambda = \frac{\left(\frac{1}{s\sqrt{2\pi}}\right)^n e^{-\frac{1}{2s^2}\sum_{i=1}^{n}(x_i-\mu_0)^2}}{\left(\frac{1}{s\sqrt{2\pi}}\right)^n e^{-\frac{1}{2s^2}\sum_{i=1}^{n}(x_i-\bar{x})^2}}$$

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$$= e^{-\frac{1}{2s^2}\left[\sum(x_i-\mu_0)^2 - \sum(x_i-\bar{x})^2\right]}$$

$$= e^{-\frac{1}{2s^2}\left[\sum(x_i^2+\mu_0^2-2\mu_0 x_i) - \sum(x_i^2+\bar{x}^2-2\bar{x}x_i)\right]}$$

$$= e^{-\frac{1}{2s^2}\left[\sum x_i^2 + n\mu_0^2 - 2\mu_0 n\bar{x} - \sum x_i^2 - n\bar{x}^2 + 2\bar{x}n\bar{x}\right]}$$

$$= e^{-\frac{1}{2s^2}\left[n\mu_0^2 - 2\mu_0 n\bar{x} + n\bar{x}^2\right]}$$

$$= e^{-\frac{1}{2s^2} n\left[\mu_0^2 - 2\mu_0\bar{x} + \bar{x}^2\right]}$$

$$= e^{-\frac{1}{2s^2} n\left[\bar{x}-\mu_0\right]^2}$$

Now $\lambda \leq \lambda_0$

$$\Rightarrow \exp\left[\frac{-n}{2s^2}(\bar{x}-\mu_0)^2\right] \leq \lambda_0$$

$$\Rightarrow \frac{-n}{2s^2}(\bar{x}-\mu_0)^2 \leq \log\lambda_0$$

$$\Rightarrow \frac{n}{s^2}(\bar{x}-\mu_0)^2 \geq -2\log\lambda_0$$

$$\Rightarrow \left(\frac{\sqrt{n}(\bar{x}-\mu_0)}{s}\right)^2 \geq -2\log\lambda_0$$

$$\left|\frac{\sqrt{n}(\bar{x}-\mu_0)}{s}\right| \geq (-2\log\lambda_0)^{1/2} = \lambda_1 \text{ (say)}$$

thus the C.R of L.R. test is,

$$W_0 = \left\{ \underset{\sim}{x} : \left| \frac{\sqrt{n}(\bar{x} - \mu_0)}{s} \right| \geq A_1 \right\}$$

where A_1 is s.t

$P_{\mu_0}(W) = \alpha$

Now, under $H_0 : \mu = \mu_0$ the

Type A region or Locally MPCR →:

Problem →:

To test $H_0: \theta = \theta_0$ against $H_1: \theta \neq \theta_0$.

Assumptions →:

Let w_0 be a critical region

(i) The power funⁿ $P_\theta(w_0)$ be twice differ-enable w.r.to θ.

(ii) The 2nd derivative of $P_\theta(w_0)$ is continuous in the neighbourhood of $\theta = \theta_0$.

Definition →:

The region w_0 is said to be a type-A region or (Locally most powerful unbiased critical region (LMPUCR) of size 'α' for testing $H_0: \theta = \theta_0$ against composite $H_1: \theta \neq \theta_0$ if

(i) $P_{\theta_0}(w_0) = \alpha$ ---(1)

(2) $\frac{d}{d\theta} P_\theta(\omega_0)\Big|_{\theta=\theta_0} = 0$ ---(2)

(3) $\frac{d^2}{d\theta^2} P_\theta(\omega_0)\Big|_{\theta=\theta_0} \geq \frac{d^2}{d\theta^2} P_\theta(\omega)\Big|_{\theta=\theta_0}$ ---(3)

whatever the other region ω satisfing condition (1) & (2) may be.

Explaination→:

Let us expand $P_\theta(\omega)$ about $\theta=\theta_0$ in a Taylor's series

$$P_\theta(\omega) = P_{\theta_0}(\omega) + (\theta-\theta_0)\frac{d}{d\theta}P_\theta(\omega)\Big|_{\theta=\theta_0} + \frac{(\theta-\theta_0)^2}{2}\frac{d^2}{d\theta^2}P_\theta(\omega)\Big|_{\theta=\theta_0} + \eta$$

where, $\eta \to 0$ as $\theta \to \theta_0$.

$$P_\theta(\omega) = \alpha + 0 + \frac{(\theta-\theta_0)^2}{2}\frac{d^2}{d\theta^2}P_\theta(\omega)\Big|_{\theta=\theta_0}$$

Similarly,

$$P_\theta(\omega_0) = \alpha + \frac{(\theta-\theta_0)^2}{2}\frac{d^2}{d\theta^2}P_\theta(\omega_0)\Big|_{\theta=\theta_0}$$

But if ω_0 is a type A region then,

$$\frac{d^2}{d\theta^2}P_\theta(\omega_0)\Big|_{\theta=\theta_0} > \frac{d^2}{d\theta^2}P_\theta(\omega)\Big|_{\theta=\theta_0}$$

or $\alpha + \frac{(\theta-\theta_0)^2}{2} P_\theta''(w_0)\Big|_{\theta=\theta_0} \geq \alpha + \frac{(\theta-\theta_0)^2}{2} P_\theta''(w)\Big|_{\theta=\theta_0}$

Thus, if $|\theta-\theta_0|$ is small

$$P_\theta(w_0) \geq P_\theta(w)$$

This shows that w_0 is most powerful critical region but only in the neighbourhood of $\theta=\theta_0$ i.e w_0 is MPCR for $H_0: \theta=\theta_0$ v/s $H_1: \theta=\theta_1$ where $|\theta_0-\theta_1|$ is small i.e these are also called locally most powerful region.

Theorem →:

Every type-A region is unbiased.

i.e $\frac{d}{d\theta} P_\theta(w_0)\Big|_{\theta=\theta_0} = 0 \Rightarrow P_\theta(w_0) \geq \alpha$.

Proof →: Expanding $P_\theta(w_0)$ in Taylor's series, we get

$P_\theta(w_0) = P_{\theta_0}(w_0) + (\theta-\theta_0)\frac{d}{d\theta} P_\theta(w_0)\Big|_{\theta=\theta_0}$

+ some +ve quantities

Now,

$\frac{d}{d\theta} P_\theta(w_0)\Big|_{\theta=\theta_0} = 0$

$\Rightarrow P_\theta(w_0) \geq \alpha +$ +ve quantity

$\Rightarrow P_\theta(\omega_0) \geq \alpha$

$\Rightarrow \omega_0$ is unbiased.

Type A_1 region $\Rightarrow$:

The condition

$$\frac{d^2}{d\theta^2} P_\theta(\omega_0)\Big|_{\theta=\theta_0} \geq \frac{d^2}{d\theta^2} P_\theta(\omega)\Big|_{\theta=\theta_0}$$

gives rise to local optimality of a type A region if we replace this condition by ① ensuring maximum power for all $\theta \neq \theta_0$ we get what is called as type A_1 is region.

Definition $\rightarrow$:

The region ω_0 is said to be A_1, a type A_1 critical region of size α for testing $H_0: \theta = \theta_0$ against $H_1: \theta \neq \theta_0$ if

(i) $P_{\theta_0}(\omega_0) = \alpha$ —— ①

(ii) $\frac{d}{d\theta} P_\theta(\omega_0)\Big|_{\theta=\theta_0} = 0$ —— ②

(iii) $P_\theta(\omega_0) \geq P_\theta(\omega) \; \forall \, \theta \neq \theta_0$ —— ③

what ever the other region ω satisfying condition ① & ② may be.

Optimum Region & Sufficient statistic:

Let $x_1, x_2 -- x_n$ be a random sample from $f(x, \theta)$ and further let T be a sufficient statistic of θ then by factorization theorem, we have

$L(\underset{\sim}{x}, \theta) = g(t, \theta) \cdot h(\underset{\sim}{x})$ --- (i)

here $g(t, \theta)$ is the funⁿ of t & θ only & $h(\underset{\sim}{x})$ is free from θ. Also note that

$L'(\underset{\sim}{x}, \theta) = g'(t, \theta) \cdot h(\underset{\sim}{x})$ --- (2)

$L''(\underset{\sim}{x}, \theta) = g''(t, \theta) \cdot h(\underset{\sim}{x})$ --- (3)

(1) To test $H_0: \theta = \theta_0$ V/S $H_1: \theta = \theta_1$ the mpCR is given by

$\omega_0 = \{ \underset{\sim}{x} : L(\underset{\sim}{x}, \theta_1) \geq k L(\underset{\sim}{x}, \theta_0) \}$

$\omega_0 = \{ \underset{\sim}{x} : g(t, \theta_1) \geq k\, g(\underset{\sim}{x}, \theta_0) \}$

thus the mpCR is the fun^n of t only.

(ii) To test $H_0: \theta = \theta_0$ V/S $H_1: \theta > \theta_0$ ($\theta < \theta_0$) if the critical region ω_0 is free from θ_1 it is UMPCR and depend on t.

(iii) To test $H_0: \theta = \theta_0$ V/S $H_1: \theta \neq \theta_0$ the type A region is given by

$$\omega = \{ \underset{\sim}{x} : L''(\underset{\sim}{x}, \theta_0) \geq K_1 L(\theta_0) + K_2 L'(\theta_0) \}$$

$$\omega = \{ \underset{\sim}{x} : g''(\underset{\sim}{x}, \theta_0) \geq K g(t, \theta_0) + K_2 g'(t, \theta_0) \}$$

& type A_1 region is given by

$$\omega = \{ x : L(x, \theta) \geq K_1 L(\theta_0) + K_2 L'(\theta_0) \}$$

$$\omega = \{ \underset{\sim}{x} : g''(t, \theta) \geq K_1 g(t, \theta_0) + K_2 g'(t, \theta_0) \}$$

thus the type A region & type A_1 region are also the fun^n of t only.

Hence the optimum region is bounded by the surface of the sufficient statistic T.

Similars Regions :-

Let $X_1, X_2 \cdots X_n$ be a random sample from $f(\underset{\sim}{x}, \underset{\sim}{\theta})$ where $\underset{\sim}{\theta} = (\theta_1, \theta_2 \cdots \theta_m)$ denotes

$\xi = (\theta_1, \theta_2 \cdots \theta_k)$

$\eta = (\theta_{k+1}, \cdots \theta_m)$

$\xi_0 = (\theta_{10}, \theta_{20}, \cdots \theta_{k0})$

$\eta_0 = (\theta_{(k+1)0}, \cdots \theta_{m0})$

Consider the testing of $H_0: \theta_1 = \theta_{10}, \theta_2 = \theta_{20}$ — $\theta_k = \theta_{k0}$ against a simple or composite alternative hypothesis. Here H_0 specify only k out of total m parameters and leavy $(m-k)$ parameters unspecified and therefore H_0 is null hypothesis with m-k degree of freedom. Here $\eta = (\theta_{k+1}, \cdots \theta_m)$ are called Luisanco parameters.

Let Θ_0 & Θ_1 be the parametric spaces under H_0 & H_1 respectively.

Definition :-

A region ω is said to be a similar region of size 'α' if

$$P[\omega | H_0] = \alpha \quad \forall\ \theta_{k+1}, \theta_{k+2}, \ldots, \theta_m$$

i.e $P[\omega | \theta_{10}, \theta_{20}, \ldots \theta_{k0}, \theta_{k+1}, \ldots \theta_m]$

$$= \alpha \ \forall\ \theta_{k+1}, \ldots \theta_m$$

Note→:

These are called similar regions since they have the similar property as that of the sample space i.e $P(S) = 1 \ \forall\ \theta_1, \ldots \theta_m$.

The similar regions can not always be constructed. They can be constructed when sufficient statistic for each of the unspecified parameters exist or when jointly sufficient statistic exist for unknown paramters.

Complete.

$$\int_0^{\lambda_0} g(\lambda \mid H_0)\, d\lambda \le \alpha \qquad \ldots(18\cdot23)$$

for all values of the parameters in Θ_0.

However, if we are dealing with large samples, a fairly satisfactory situation to this testing of hypothesis problem exists as stated (without proof) in the following theorem.

Theorem 18·4. *Let $x_1, x_2, \ldots, x_n$ be a random sample from a population with p.d.f. $f(x\,;\theta_1, \theta_2, \ldots, \theta_k)$ where the parameter space Θ is k-dimensional. Suppose we want to test the composite hypothesis*

$$H_0 : \theta_1 = \theta_1', \theta_2 = \theta_2', \ldots, \theta_r = \theta_r' ; r < k$$

where $\theta_1', \theta_2', \ldots, \theta_r'$ are specified numbers. When H_0 is true, $-2 \log_e \lambda$ is asymptotically distributed as chi-square with r degrees of freedom, i.e., under H_0,

$$-2 \log \lambda \sim \chi_{(r)}^2\,, \quad \text{if } n \text{ is large.} \qquad \ldots(18\cdot24)$$

Since $0 \le \lambda \le 1$, $-2 \log_e \lambda$ is an increasing function of λ and approaches infinity when $\lambda \to 0$, the critical region for $-2 \log \lambda$ being the right hand tail of the chi-square distribution. Thus at the level of significance 'α', the test may be given as follows :

$$\text{Reject } H_0 \text{ if } -2 \log_e \lambda > \chi_r^2(\alpha)$$

where $\chi_r^2(\alpha)$ is the upper α-point of the chi-square distribution with r d.f. given by :

$$P[\chi^2 > \chi_{(r)}^2(\alpha)] = \alpha,$$

otherwise H_0 may be accepted.

18·6·1. Properties of Likelihood Ratio Test. Likelihood ratio (*L.R.*) test principle is an intuitive one. If we are testing a simple hypothesis H_0 against a simple alternative hypothesis H_1 then the *LR* principle leads to the same test as given by the Neyman-Pearson lemma. This suggests that *LR* test has some desirable properties, specially large sample properties.

In *LR* test, the probability of type *I* error is controlled by suitably choosing the cut off point λ_0. *LR* test is generally *UMP* if an *UMP* test at all exists. We state below, the two asymptotic properties of *LR* tests.

1. Under certain conditions, $-2 \log_e \lambda$ has an asymptotic chi-square distribution.

2. Under certain assumptions, *LR* test is consistent.

Now we shall illustrate how the likelihood ratio criterion can be used to obtain various standard tests of significance in Statistics.

18·6·2. Test For the Mean of a Normal Population. Let us take the problem of testing if the mean of a normal population has a specified value. Let $(x_1, x_2, \ldots, x_n)$ be a random sample of size n from the normal population with mean μ and variance σ^2, where μ and σ^2 are unknown. Suppose we want to test the (composite) null hypothesis : $H_0 : \mu = \mu_0$ (specified), $0 < \sigma^2 < \infty$

against the composite alternative hypothesis : $H_1 : \mu \ne \mu_0 ; 0 < \sigma^2 < \infty$

In this case the parameter space Θ is given by

$$\Theta = \{(\mu, \sigma^2) : -\infty < \mu < \infty,\ 0 < \sigma^2 < \infty\}$$

and the subspace Θ_0 determined by the null hypothesis H_0 is given by

$$\Theta_0 = \{(\mu, \sigma^2) : \mu = \mu_0,\ 0 < \sigma^2 < \infty\}$$

The likelihood function of the sample observations $x_1, x_2, \ldots, x_n$ is given by

$$L = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} . \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\} \qquad \ldots(18\cdot25)$$

The maximum likelihood estimates of μ and σ^2 are given by :

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}, \qquad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 = s^2 \qquad \ldots(18\cdot26)$$

Hence, substituting in (18·25), the maximum of L in the parameter space Θ is given by

$$L(\hat{\Theta}) = \left(\frac{1}{2\pi s^2}\right)^{n/2} . \exp(-n/2) \qquad \ldots(18\cdot27)$$

In Θ_0, the only variate parameter is σ^2 and MLE of σ^2 for given $\mu = \mu_0$ is given by

$$\hat{\sigma}^2 = \frac{1}{n}\Sigma(x_i-\mu_0)^2 = s_0^2, \text{ (say)} \qquad \ldots(18\cdot28)$$

$$= \frac{1}{n}\Sigma(x_i-\bar{x}+\bar{x}-\mu_0)^2$$

$$= \frac{1}{n}\Sigma(x_i-\bar{x})^2 + (\bar{x}-\mu_0)^2,$$

the product term vahishes, since $\Sigma(x_i-\bar{x})(\bar{x}-\mu_0) = (\bar{x}-\mu_0)\Sigma(x_i-\bar{x}) = 0$

$$\therefore \qquad \hat{\sigma}^2 = s^2 + (\bar{x}-\mu_0)^2 = s_0^2, \text{ (say)}. \qquad \ldots(18\cdot28a)$$

Hence, substituting in (18·25),

$$L(\hat{\Theta}_0) = \left(\frac{1}{2\pi s_0^2}\right)^{n/2} \exp(-n/2) \qquad \ldots(18\cdot28b)$$

The ratio of (18·28b) and (18·27) gives the likelihood ratio criterion

$$\lambda = \frac{L(\hat{\Theta}_0)}{L(\hat{\Theta})} = \left(\frac{s^2}{s_0^2}\right)^{n/2} \qquad \ldots(18\cdot29)$$

$$= \left\{\frac{s^2}{s^2+(\bar{x}-\mu_0)^2}\right\}^{n/2} = \left\{\frac{1}{1+[(\bar{x}-\mu_0)^2/s^2]}\right\}^{n/2} \quad \text{[From 18·28(a)]} \quad \ldots(18\cdot29a)$$

We have proved earlier (§ 16·2) that under H_0, the statistic

$$t = \frac{\bar{x}-\mu_0}{S/\sqrt{n}}, \text{ where } S^2 = \frac{1}{n-1}\Sigma(x_i-\bar{x})^2 = \frac{ns^2}{n-1}$$

follows Student's t-distribution with $(n-1)$ $d.f.$

$$\text{Thus,} \quad t = \frac{\bar{x}-\mu_0}{S/\sqrt{n}} = \frac{\bar{x}-\mu_0}{s/\sqrt{n-1}} \sim t_{n-1} \qquad \ldots(18\cdot$$

Substituting in (18·29a), we get

$$\lambda = \frac{1}{\left(1+\dfrac{t^2}{n-1}\right)^{n/2}} = \phi(t^2), \text{ (say)} \qquad \ldots(18\cdot$$

The likelihood ratio test for testing H_0 against H_1 consists in finding a critical region of the type $0 < \lambda < \lambda_0$, where λ_0 is given by (18·21a), which requires the distribution of λ under H_0. In this case, it is not necessary to obtain the distribution of λ since $\lambda = \phi(t)$ is a monotonic function of t^2 and the test can well be carried on with t^2 as a criterion as with λ [c.f. Theorem 18·1]. Now $t^2 = 0$ when $\lambda = 1$ and t^2 becomes infinite when $\lambda = 0$. The critical region of the LR test *viz.*, $0 < \lambda < \lambda_0$, on using (18·31) is equivalent to

$$\left(1 + \frac{t^2}{n-1}\right)^{-n/2} \le \lambda_0 \qquad \Rightarrow \qquad \left(1 + \frac{t^2}{n-1}\right)^{n/2} \ge \lambda_0^{-1}$$

$$\Rightarrow \qquad \frac{t^2}{n-1} \ge (\lambda_0)^{-2/n} - 1 \qquad \Rightarrow \qquad t^2 \ge (n-1)\,[\lambda_0^{-2/n} - 1] = A^2, \text{ (say)}.$$

Thus the critical region may well be defined by

$$|\,t\,| = \left|\frac{\sqrt{n}\,(\bar{x} - \mu_0)}{S}\right| \ge A \qquad \ldots(18\cdot32)$$

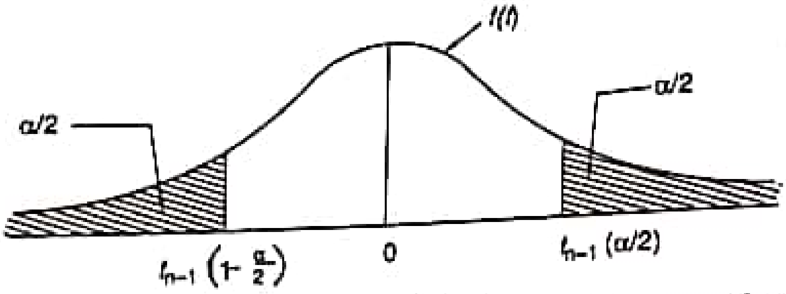
where the constant A is determined such that

$$P[\,|\,t\,| \ge A \mid H_0] = \alpha \qquad \ldots(18\cdot33)$$

Since under H_0, the statistic t follows Student's t distribution with $(n-1)$ *d.f.*, $A = t_{n-1}(\alpha/2)$ where the symbol $t_n(\alpha)$ stands for the *right tail* 100 α% point of the t-distribution with n d.f. given by :

$$P\{t > t_n(\alpha)\} = \int_{t_n(\alpha)}^{\infty} f(t)\,dt = \alpha, \qquad \ldots(18\cdot33a)$$

where $f(\cdot)$ is the p.d.f. of Student's t with n d.f. The critical region is shown in the following diagram.



Thus for testing $H_0 : \mu = \mu_0$ against $\mu \ne \mu_0$ (σ^2-unknown), we have the two-tailed t-test defined as follows :

$$\textit{If} \quad |\,t\,| = \left|\frac{\sqrt{n}\,(\bar{x} - \mu_0)}{S}\right| > t_{n-1}(\alpha/2), \textit{ reject } H_0$$

and if $|\,t\,| < t_{n-1}(\alpha/2)$, H_0 may be accepted.

Important Remarks 1. Let us now consider the problem of testing the hypothesis :

$$H_0 : \mu = \mu_0,\ 0 < \sigma^2 < \infty\,.$$

against the alternative hypothesis

$$H_1 : \mu > \mu_0,\ 0 < \sigma^2 < \infty$$

Here $\qquad \Theta = \{(\mu, \sigma^2) : -\infty < \mu < \infty,\ 0 < \sigma^2 < \infty\}$

and $\qquad \Theta_0 = \{(\mu, \sigma^2) : \mu = \mu_0,\ 0 < \sigma^2 < \infty\}$

The maximum likelihood estimates of μ and σ^2 belonging to Θ are given by

$$\hat{\mu} = \begin{cases} \bar{x}, & \text{if } \bar{x} \geq \mu_0 \\ \mu_0, & \text{if } \bar{x} < \mu_0 \end{cases} \quad \ldots(18\cdot 3$$

and

$$\hat{\sigma}^2 = \begin{cases} s^2, & \text{if } \bar{x} \geq \mu_0 \\ s_0^2, & \text{if } \bar{x} < \mu_0 \end{cases} \quad \ldots(18\cdot 34$$

where

$$s_0^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu_0)^2 \quad \ldots(18\cdot 34$$

Thus

$$L(\hat{\Theta}) = \begin{cases} \left(\frac{1}{2\pi s^2}\right)^{n/2} . \exp(-n/2), & \text{if } \bar{x} \geq \mu_0 \\ \left(\frac{1}{2\pi s_0^2}\right)^{n/2} . \exp(-n/2), & \text{if } \bar{x} < \mu_0 \end{cases} \quad \ldots(18\cdot 35$$

In Θ_0, the only unknown parameter is σ^2 whose *MLE* is given by $\hat{\sigma}^2 = s_0^2$. Thus

$$L(\hat{\Theta}_0) = \left(\frac{1}{2\pi s_0^2}\right)^{n/2} . \exp(-n/2) \quad \ldots(18\cdot 36$$

$$\therefore \quad \lambda = \frac{L(\hat{\Theta}_0)}{L(\hat{\Theta})} = \begin{cases} (s^2/s_0^2)^{n/2}, & \text{if } \bar{x} \geq \mu_0 \\ 1, & \text{if } \bar{x} < \mu_0 \end{cases} \quad \ldots(18\cdot 37$$

Thus the sample observations $(x_1, x_2, \ldots, x_n)$ for which $\bar{x} < \mu_0$ are to be included in the acceptance region. Hence for the sample observations for which $\bar{x} \geq \mu_0$, the likelihood ratio criterion becomes

$$\lambda = (s^2/s_0^2)^{n/2}, \ \bar{x} \geq \mu_0 \quad \ldots(18\cdot 37a$$

which is the same as the expression obtained in (18·29). Proceeding similarly as in the above problem, the critical region of the form $0 < \lambda < \lambda_0$ will be equivalently given by [c.f. (18·32)]

$$t^2 = \frac{n(\bar{x} - \mu_0)^2}{S^2} \geq A^2 \ \text{ or by } \ t = \frac{\sqrt{n}(\bar{x} - \mu_0)}{S} \geq A \qquad (\because \bar{x} \geq \mu_0) \quad \ldots(18\cdot 38$$

where t follows Student's t distribution with $(n-1)$ d.f. The constant A is to be determined so that

$$P(t > A) = \alpha \quad \Rightarrow \quad A = t_{n-1}(\alpha) \quad \ldots(18\cdot 39$$

Hence for testing $H_0 : \mu = \mu_0$ against $H_1 : \mu > \mu_0$, we have the right tailed-t-test defined as follows :

Reject H_0 if $t = \frac{\sqrt{n}(\bar{x} - \mu_0)}{S} > t_{n-1}(\alpha)$ and if $t < t_{n-1}(\alpha)$, H_0 may be accepted.

2. If we want to test $\qquad H_0 : \mu = \mu_0, \ 0 < \sigma^2 < \infty$

against the alternative hypothesis : $\qquad H_1 : \mu < \mu_0, \ 0 < \sigma^2 < \infty,$

then proceeding exactly similarly as in Remark 1 above, we shall get the critical region given by :

$$t < -t_{n-1}(\alpha) \quad \ldots(18\cdot 40$$

S. No.	Alternative Hypothesis	Test	Test Statistic	Critical region at level of significance 'α'	$(1-\alpha)$ confidence coefficient for $\frac{\sigma_1^2}{\sigma_2^2}$
1.	$\frac{\sigma_1^2}{\sigma_2^2} > \delta_0^2$	Right-tailed	$F = \frac{S_1^2}{S_2^2} \cdot \frac{1}{\delta_0^2}$	$F > F_{m-1,n-1}(\alpha)$	$\frac{\sigma_1^2}{\sigma_2^2} \ge \frac{S_1^2}{S_2^2} \times \frac{1}{F_{m-1,n-1}(\alpha)}$
2.	$\frac{\sigma_1^2}{\sigma_2^2} < \delta_0^2$	Left-tailed	— do —	$F < F_{m-1,n-1}(1-\alpha)$	$\frac{\sigma_1^2}{\sigma_2^2} \le \frac{S_1^2}{S_2^2} \times \frac{1}{F_{m-1,n-1}(1-\alpha)}$
3.	$\frac{\sigma_1^2}{\sigma_2^2} \ne \delta_0^2$	Two-tailed	— do —	$F > F_{m-1,n-1}(\alpha/2)$ and $F < F_{m-1,n-1}(1-\alpha/2)$	$\frac{S_1^2}{S_2^2} \cdot \frac{1}{F_{m-1,n-1}(\alpha/2)} \le \frac{\sigma_1^2}{\sigma_2^2} \le \frac{S_1^2}{S_2^2} \cdot \frac{1}{F_{m-1,n-1}(1-\alpha/2)}$

18·6·7. Test for the Equality of Variances of Several Normal Populations. Let X_{ij}, $(j = 1, 2, \ldots, n_i)$ be a random sample of size n_i from the normal population $N(\mu_i, \sigma_i^2)$, $i = 1, 2, \ldots, k$. We want to test the null hypothesis :

$H_0 : \sigma_1^2 = \sigma_2^2 = \ldots = \sigma_k^2 = \sigma^2$ (unspecified), with $\mu_1, \mu_2, \ldots, \mu_k$ (unspecified), against the alternative hypothesis :

$H_1 : \sigma_i^2$ $(i, 2, \ldots, k)$, are not all equal; $\mu_1, \mu_2, \ldots, \mu_k$ (unspecified).

Here we have

$$\Theta = \{\mu_1, \mu_2, \ldots, \mu_k\ ;\ \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2) : -\infty < \mu_i < \infty,\ \sigma_i^2 > 0\ (i = 1, 2, \ldots, k)\}$$

and

$$\Theta_0 = \{\mu_1, \mu_2, \ldots, \mu_k\ ;\ \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2) : -\infty < \mu_i < \infty, \sigma_i^2 = \sigma^2 > 0, (i = 1, 2, \ldots, k)\}$$

The likelihood function of the sample observations x_{ij}, $(j = 1, 2, \ldots, n_i : i = 1, 2, \ldots, k)$ is given by

$$L = \prod_{i=1}^{k} \left[\left(\frac{1}{2\pi\sigma_i^2} \right)^{n_i/2} \cdot \exp\left\{ - \frac{1}{2\sigma_i^2} \sum_{j=1}^{n_i} (x_{ij} - \mu_i) \right\} \right] \quad \ldots(18\cdot86)$$

It can be easily seen that in Θ the MLE's of μ_i's and σ_i's are given by

$$\hat{\mu}_i = \text{and } \hat{\sigma}_i^2 = \frac{1}{n_i} \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2 = s_i^2 \quad \ldots(18\cdot87)$$

$$\therefore \quad L(\hat{\Theta}) = \prod_{i=1}^{k} \left\{ \left(\frac{1}{2\pi s_i^2} \right)^{n_i/2} \cdot \exp\left(- \frac{n_i}{2} \right) \right\}$$

$$= \exp\left(-\frac{n}{2} \right) \cdot \prod_{i=1}^{k} \left\{ \left(\frac{1}{2\pi s_i^2} \right)^{n_i/2} \right\}, \text{ where } n = \Sigma n_i \quad \ldots(18\cdot88)$$

$$\left.\begin{aligned}
&\frac{\partial}{\partial \mu_1}\log L = 0 \quad \Rightarrow \quad \hat{\mu}_1 = \frac{1}{m}\sum_{i=1}^{m} x_{1i} = \bar{x}_1\\
&\frac{\partial}{\partial \mu_2}\log L = 0 \quad \Rightarrow \quad \mu_2 = \frac{1}{n}\sum_{j=1}^{n} x_{2j} = \bar{x}_2\\
&\frac{\partial}{\partial \sigma_1^2}\log L = 0 \quad \Rightarrow \quad \hat{\sigma}_1^2 = \frac{1}{m}\sum_{i=1}^{m} (x_{1i} - \bar{x}_1)^2 = s_1^2, \text{ (say).}\\
\text{and}\quad &\frac{\partial}{\partial \sigma_2^2}\log L = 0 \quad \Rightarrow \quad \hat{\sigma}_2^2 = \frac{1}{n}\sum_{j=1}^{n} (x_{2j} - \bar{x}_2)^2 = s_2^2, \text{ (say).}
\end{aligned}\right\} \qquad \text{...(18·41}a)$$

Substituting in (16·41), we get

$$L(\hat{\Theta}) = \left(\frac{1}{2\pi s_1^2}\right)^{m/2} \cdot \left(\frac{1}{2\pi s_2^2}\right)^{n/2} \cdot e^{-(m+n)/2} \qquad \text{...(18·42)}$$

In Θ_0, we have $\mu_1 = \mu_2 = \mu$ and the likelihood function is given by :

$$L(\Theta_0) = \left(\frac{1}{2\pi\sigma_1^2}\right)^{m/2} \cdot \exp\left\{-\frac{1}{2\sigma_1^2}\sum_{i=1}^{m}(x_{1i}-\mu)^2\right\} \times \left(\frac{1}{2\pi\sigma_2^2}\right)^{n/2} \cdot \exp\left\{-\frac{1}{2\sigma_2^2}\sum_{j=1}^{n}(x_{2j}-\mu)^2\right\}$$

To obtain the maximum value of $L(\Theta_0)$ for variations in μ, σ_1^2 and σ_2^2, it will be seen that estimate of μ is obtained as the root of a cubic equation

$$\frac{m^2(\bar{x}_1 - \mu)}{\sum\limits_{i=1}^{m}(x_{1i} - \hat{\mu})^2} + \frac{n^2(\bar{x}_2 - \mu)}{\sum\limits_{j=1}^{n}(x_{2j} - \hat{\mu})^2} \qquad \text{...(18·43)}$$

and is thus a complicated function of the sample observations. Consequently the likelihood ratio criterion λ will be a complex function of the observations and its distribution is quite tedious since it involves the ratio of two variances. Consequently, it is impossible to obtain the critical region $0 < \lambda < \lambda_0$ for given α since the distribution of the population variances is ordinarily unknown. However, in any given instance the cubic equation (16·43) can be solved for μ by numerical analysis technique and thus λ can be computed. Finally, as an approximate test, $-2 \log_e \lambda$ can be regarded as a χ^2-variate with 1 *d.f.* (*c.f. Theorem 18·2*).

Case 2. ***Population Variances are equal, i.e.,*** $\sigma_1^2 = \sigma_2^2 = \sigma^2$, (say). In this case

$$\Theta = \{(\mu_1, \mu_2, \sigma^2) : -\infty < \mu_i < \infty, \sigma^2 > 0, (i = 1, 2)\}$$

$$\Theta_0 = \{(\mu, \sigma^2) : -\infty < \mu < \infty, \sigma^2 > 0\}$$

The likelihood function is then given by

$$L = \left(\frac{1}{2\pi\sigma^2}\right)^{(m+n)/2} \cdot \exp\left[-\frac{1}{2\sigma^2}\left\{\sum_{i=1}^{m}(x_{1i}-\mu_1)^2 + \sum_{j=1}^{n}(x_{2j}-\mu_2)^2\right\}\right] \qquad \text{...(18·44)}$$

For $\mu_1, \mu_2, \sigma^2 \in \Theta$, the maximum likelihood equations are given by

$$\frac{\partial}{\partial \mu_1}\log L = 0 \quad \Rightarrow \quad \hat{\mu}_1 = \bar{x}_1 \quad \text{and} \quad \frac{\partial}{\partial \mu_2}\log L = 0 \quad \Rightarrow \quad \hat{\mu}_2 = \bar{x}_2 \qquad \text{...(18·45)}$$

and $$\frac{\partial}{\partial \sigma^2}\log L = 0 \quad \Rightarrow \quad \hat{\sigma}^2 = \frac{1}{m+n}\left\{\Sigma(x_{1i} - \hat{\mu}_1)^2 + \Sigma(x_{2j} - \hat{\mu}_2)^2\right\}$$

$$\Rightarrow \quad \hat{\sigma}^2 = \frac{1}{m+n}\left\{\Sigma(x_{1i}-\bar{x}_1)^2 + \Sigma(x_{2j}-\bar{x}_2)^2\right\} = \frac{1}{m+n}\left(ms_1^2 + ns_2^2\right) \qquad \dots(18\cdot45a)$$

Substituting the values from (18·45) and (18·45a) in (18·44), we get

$$L(\hat{\Theta}) = \left\{\frac{(m+n)}{2\pi(ms_1^2+ns_2^2)}\right\}^{(m+n)/2} . \exp\left\{-\frac{1}{2}(m+n)\right\} \qquad \dots(18\cdot46)$$

In Θ_0, $\mu_1 = \mu_2 = \mu$ (say) and we get

$$L(\Theta_0) = \left(\frac{1}{2\pi\sigma^2}\right)^{(m+n)/2} . \exp\left[-\frac{1}{2\sigma^2}\left\{\sum_{i=1}^{m}(x_{1i}-\mu)^2 + \sum_{j=1}^{n}(x_{2j}-\mu)^2\right\}\right] \qquad \dots(18\cdot47)$$

$$\Rightarrow \quad \log L(\Theta_0) = C - \frac{m+n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\left\{\sum_i(x_{1i}-\mu)^2 + \sum_j(x_{2j}-\mu)^2\right\},$$

where C is a constant independent of μ and σ^2.

The likelihood equation for estimating μ gives

$$\frac{\partial}{\partial\mu}\log L = -\frac{1}{\sigma^2}\left\{\sum_{i=1}^{m}(x_{1i}-\mu) + \sum_{j=1}^{n}(x_{2j}-\mu)\right\} = 0 \quad \Rightarrow \quad (m\bar{x}_1 + n\bar{x}_2) - (m+n)\mu = 0$$

$$\Rightarrow \quad \hat{\mu} = \frac{1}{m+n}\left[m\bar{x}_1 + n\bar{x}_2\right] \qquad \dots(18\cdot48)$$

$$\text{Also} \quad \frac{\partial}{\partial\sigma^2}\log L = 0 \quad \Rightarrow \quad -\frac{(m+n)}{2\sigma^2} + \frac{1}{2\sigma^4}\left[\Sigma(x_{1i}-\mu)^2 + \Sigma(x_{2j}-\mu)^2\right] = 0$$

$$\Rightarrow \quad \hat{\sigma}^2 = \frac{1}{m+n}\left\{\Sigma(x_{1i}-\hat{\mu})^2 + \Sigma(x_{2j}-\hat{\mu})^2\right\} \qquad \dots(18\cdot49)$$

$$\text{But} \quad \sum_{i=1}^{m}(x_{1i}-\hat{\mu})^2 = \sum_{i=1}^{m}(x_{1i}-\bar{x}_1+\bar{x}_1-\hat{\mu})^2$$

$$= \Sigma(x_{1i}-\bar{x}_1)^2 + m(\bar{x}_1-\hat{\mu})^2,$$

the product term vanishes since $\sum_i (x_{1i}-\bar{x}_1) = 0$.

$$\therefore \quad \sum_{i=1}^{m}(x_{1i}-\hat{\mu})^2 = ms_1^2 + m\left(\bar{x} - \frac{m\bar{x}_1 + n\bar{x}_2}{m+n}\right)^2$$

$$= ms_1^2 + \frac{mn^2(\bar{x}_1-\bar{x}_2)^2}{(m+n)^2}$$

Similarly, we shall get :

$$\sum_{j=1}^{n}(x_{2j}-\hat{\mu})^2 = ns_2^2 + \frac{nm^2(\bar{x}_2-\bar{x}_1)^2}{(m+n)^2}$$

Substituting in (18·49), we get

$$\hat{\sigma}^2 = \frac{1}{m+n}\left\{ms_1^2 + ns_2^2 + \frac{mn}{m+n}(\bar{x}_1-\bar{x}_2)^2\right\} \qquad \dots(18\cdot49a)$$

Substituting from (18·48) and (18·49a) in (18·47), we get

$$L(\hat{\Theta}_0) = \left\{\frac{(m+n)}{2\pi\left(ms_1^2 + ns_2^2 + \frac{mn}{m+n}(\bar{x}_1-\bar{x}_2)^2\right)}\right\}^{(m+n)/2} \times \exp\left(-\frac{m+n}{2}\right) \qquad \dots(18\cdot50)$$

$$\therefore \qquad \lambda = \frac{L(\hat{\Theta}_0)}{L(\hat{\Theta})} = \left\{ \frac{ms_1^2 + ns_2^2}{ms_1^2 + ns_2^2 + \frac{mn}{m+n}(\bar{x}_1 - \bar{x}_2)^2} \right\}^{(m+n)/2}$$

$$= \left[\frac{1}{\left\{ 1 + \frac{mn(\bar{x}_1 - \bar{x}_2)^2}{(m+n)(ms_1^2 + ns_2^2)} \right\}} \right]^{(m+n)/2} \qquad \ldots(18\cdot51)$$

We know that (c.f. § 16·3·3), under the null hypothesis $H_0 : \mu_1 = \mu_2$, the statistic :

$$t = \frac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\frac{1}{m} + \frac{1}{n}}}, \qquad \ldots(18\cdot52)$$

where $$S^2 = \frac{1}{m+n-2}(ms_1^2 + ns_2^2) \qquad \ldots(18\cdot52a)$$

follows Student's t-distribution with $(m + n - 2)$ d.f. Thus in terms of t, we get

$$\lambda = \left(1 + \frac{t^2}{m+n-2}\right)^{-(m+n)/2} \qquad \ldots(18\cdot53)$$

As in § 18·7·1, the test can as well be carried with t rather than with λ. The critical region $0 < \lambda < \lambda_0$ transforms to the critical region of the type

$$t^2 > (m+n-2)\left[\frac{1}{\lambda_0^{2/(m+n)}} - 1 \right] = A^2, \text{ (say)}$$

i.e., by $$|\,t\,| > A,$$

where A is determined so that : $$P[\,|\,t\,| > A \mid H_0] = \alpha \qquad \ldots(18\cdot55)$$

Since under H_0, the statistic t follows Student's t-distribution with $(m + n - 2)$ d.f., we get from (18·55) : $A = t_{m+n-2}(\alpha/2)$ ...(18·56)

where, $t_n(\alpha)$ is the right 100α% point of the t-distribution with n $d.f.$

Thus for testing the null hypothesis $H_0 : \mu_1 = \mu_2 \,;\, \sigma_1^2 = \sigma_2^2 = \sigma^2 > 0$

against the alternative : $H_1 : \mu_1 \neq \mu_2,\, \sigma_1^2 = \sigma_2^2 = \sigma^2 > 0,$

we have the two-tailed t-test defined as follows ;

If $|\,t\,| = \left| \dfrac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\frac{1}{m} + \frac{1}{n}}} \right| > t_{m+n-2}(\alpha/2)$, *reject* H_0, *otherwise* H_0 *may be accepted.*

Remarks 1. Proceeding similarly as in Remarks to § 18·7·1, we can obtain the critical regions for testing

$$H_0 : \mu_1 = \mu_2 \,;\; \sigma_1^2 = \sigma_2^2 = \sigma^2 > 0$$

against the alternative hypothesis

$$H_1 : \mu_1 > \mu_2 \,;\; \sigma_1^2 = \sigma_2^2 = \sigma^2 > 0$$

or $$H_1' : \mu_1 < \mu_2 \,;\; \sigma_1^2 = \sigma_2^2 = \sigma^2 > 0$$

We give below, in a tabular form the critical region, the test statistic and the confidence interval for testing the hypothesis

$$H_0 : \delta = \mu_1 - \mu_2 = \delta_0, \text{ (say)},$$

against various alternatives, *viz.*, $\delta > \delta_0$, $\delta < \delta_0$ or $\delta \neq \delta_0$.

S. No.	Alternative Hypothesis	Test	Test statistic	Reject H_0 at level of significance α if	$(1-\alpha)$ Confidence interval of δ
1.	$\delta > \delta_0$	Right, tailed	$t = \dfrac{(\bar{x}_1 - \bar{x}_2) - \delta_0}{S\sqrt{\dfrac{1}{m}+\dfrac{1}{n}}}$	$t > t_{m+n-2}(\alpha) = t_1$, (say)	$\delta \geq (\bar{x}_1 - \bar{x}_2) - t_1 S\sqrt{\dfrac{1}{m}+\dfrac{1}{n}}$
2.	$\delta \neq \delta_0$	Two tailed	— do —	$\lvert t \rvert > t_{m+n-2}(\alpha/2) = t_2$, (say)	$(\bar{x}_1 - \bar{x}_2) - t_2 S\sqrt{\dfrac{1}{m}+\dfrac{1}{n}} \leq \delta \leq (\bar{x}_1 - \bar{x}_2) + t_2 S\sqrt{\dfrac{1}{m}+\dfrac{1}{n}}$

18·6·4. Test For the Equality of Means of Several Normal Populations. Let X_{ij}, ($j = 1, 2, \ldots, n_i$; $i = 1, 2, \ldots k$) be k independent random samples from k normal populations with means $\mu_1, \mu_2, \ldots, \mu_k$ respectively and unknown but common variance σ^2. In other words, the k normal populations are supposed to be *homoscedastic*. We want to test the null hypothesis

$$H_0 : \mu_1 = \mu_2 = \ldots = \mu_k = \mu \text{ (say), (unspecified)}$$
$$\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_k^2 = \sigma^2 \text{ (say), (unspecified)}$$

against the alternative hypothesis

$$H_1 : \mu_i\text{'s are not all equal, and}$$
$$\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_k^2 = \sigma^2, \text{ (unspecified)}$$

Thus we have

$$\Theta = \{(\mu_1, \mu_2, \ldots, \mu_k, \sigma^2) : -\infty < \mu_i < \infty, (i = 1, 2, \ldots, k) : \sigma^2 > 0\}$$

and $\quad \Theta_0 = \{(\mu_1, \mu_2, \ldots, \mu_k, \sigma^2) - \infty < \mu_i = \mu < \infty, (i = 1, 2, \ldots, k) \ \sigma^2 > 0\}$

The likelihood function of the sample observations is given by

$$L(\Theta) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} . \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij} - \mu_i)^2\right\} \text{ where } n = \sum_{i=1}^{k} n_i. \quad \ldots(18{\cdot}57)$$

For variations of μ_i, $(i = 1,2, \ldots, k)$ and σ^2 in Θ, the maximum likelihood estimates are given by

$$\frac{\partial}{\partial\mu_i}\log L(\Theta) = 0 \;\Rightarrow\; \sum_j (x_{ij} - \mu_i) = 0 \;\Rightarrow\; \hat{\mu}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} x_{ij} = \bar{x}_i \quad \ldots(18{\cdot}58)$$

$$\frac{\partial}{\partial\sigma^2}\log L(\Theta) = 0 \;\Rightarrow\; \hat{\sigma}^2 = \frac{1}{n}\sum_i\sum_j (x_{ij} - \hat{\mu}_i)^2$$

$$\Rightarrow \quad \hat{\sigma}^2 = \frac{1}{n}\sum_i\sum_j (x_{ij} - \bar{x}_i)^2 = \frac{S_W}{n}, \text{ (say)}, \quad \ldots(18{\cdot}58a)$$

2. For testing $H_0 : \delta = \delta_0$ against the alternative $H_1 : \delta < \delta_0$, the roles of x_1 and x_2 are interchanged and the case 1 of the table is applied.

3. If $\delta_0 = 0$, the above test reduces to testing $H_0 : \mu_1 = \mu_2$, *i.e.*, the equality of two population means.

4. If the two population variances are not equal, then for testing $H_0 : \delta = \delta_0$, we use Fisher-Behrens' d-test.

S. No.	Alternativ Hypothesis	Test	Test statistic	Reject H_0 at level of significance α if	$(1-\alpha)$ Confidence interval of δ)
1.	$\delta > \delta_0$	Right, tailed	$t = \dfrac{(\bar{x}_1 - \bar{x}_2) - \delta_0}{S\sqrt{\frac{1}{m}+\frac{1}{n}}}$	$t > t_{m+n-2}(\alpha) = t_1$, (say)	$\delta \geq (\bar{x}_1 - \bar{x}_2) - t_1 S\sqrt{\frac{1}{m}+\frac{1}{n}}$
2.	$\delta \neq \delta_0$	Two tailed	—do—	$\lvert t \rvert > t_{m+n-2}(\alpha/2) = t_2$, (say)	$(\bar{x}_1 - \bar{x}_2) - t_2 S\sqrt{\frac{1}{m}+\frac{1}{n}} \leq \delta \leq (\bar{x}_1 - \bar{x}_2) + t_2 S\sqrt{\frac{1}{m}+\frac{1}{n}}$

18·6·4. Test For the Equality of Means of Several Normal Populations. Let X_{ij}, $(j = 1, 2, \ldots, n_i;\ i = 1, 2, \ldots k)$ be k independent random samples from k normal populations with means $\mu_1, \mu_2, \ldots, \mu_k$ respectively and unknown but common variance σ^2. In other words, the k normal populations are supposed to be *homoscedastic*. We want to test the null hypothesis

$$H_0 : \mu_1 = \mu_2 = \ldots = \mu_k = \mu \text{ (say), (unspecified)}$$
$$\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_k^2 = \sigma^2 \text{ (say), (unspecified)}$$

against the alternative hypothesis

$$H_1 : \mu_i\text{'s are not all equal, and}$$
$$\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_k^2 = \sigma^2, \text{ (unspecified)}$$

Thus we have

$$\Theta = \{(\mu_1, \mu_2, \ldots, \mu_k, \sigma^2) : -\infty < \mu_i < \infty, (i = 1, 2, \ldots, k) : \sigma^2 > 0\}$$

and $$\Theta_0 = \{(\mu_1, \mu_2, \ldots, \mu_k, \sigma^2) - \infty < \mu_i = \mu < \infty, (i = 1, 2, \ldots, k)\ \sigma^2 > 0\}$$

The likelihood function of the sample observations is given by

$$L(\Theta) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \cdot \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij}-\mu_i)^2\right\} \text{ where } n = \sum_{i=1}^{k} n_i. \quad \ldots(18\cdot57)$$

For variations of μ_i, $(i = 1,2, \ldots, k)$ and σ^2 in Θ, the maximum likelihood estimates are given by

$$\frac{\partial}{\partial\mu_i}\log L(\Theta) = 0 \quad\Rightarrow\quad \sum_j (x_{ij}-\mu_i) = 0 \quad\Rightarrow\quad \hat{\mu}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} x_{ij} = \bar{x}_i \quad \ldots(18\cdot58)$$

$$\frac{\partial}{\partial\sigma^2}\log L(\Theta) = 0 \quad\Rightarrow\quad \hat{\sigma}^2 = \frac{1}{n}\sum_i\sum_j (x_{ij}-\hat{\mu}_i)^2$$

$$\Rightarrow \quad \hat{\sigma}^2 = \frac{1}{n}\sum_i\sum_j (x_{ij}-\bar{x}_i)^2 = \frac{S_W}{n}, \text{ (say)}, \quad \ldots(18\cdot58a)$$