Operation Research

Quadratic Programming Problem

(After the Kuhn-Tucker Conditions).

Wolfe's Modified Simplex Method :-

The wolfe's method for solving a quadratic programming problem can be summarized as:-

Step 1:- Introduce artifical variable A_j in Khun-Tucker conditions

$$c_j - \sum_{K=1}^{n} x_k d_{jK} - d_i a_{ij} + u_j + A_j = 0$$

for a starting feasible solution we shall have

$$x_j = 0 \; ; \quad u_j = 0 \quad , \; a_j = -c_j \quad \& \quad s_i^2 = b_i$$

Step 2:- Apply phase I^{st} of simplex method to check the feasibility of constraint eq^n $AX \leq b$. If their is no feasible sol^n terminate the sol^n procedure otherwise get an initial basic feasible solution for phase 2. To obtain the desired feasible sol^n for the following problem.

$$\text{Min } Z = \sum_{j=1}^{n} A_j$$

$$\sum_{K=1}^{n} x_k d_{jk} - d_i a_{ij} + u_j + A_j = -c_j$$

$$\sum_{j=1}^{n} a_{ij} x_j + s_i^2 = b_i$$

$$\left.\begin{array}{l} d_i S_i = 0 \\ u_i x_i = 0 \end{array}\right\} \text{these are known as Complementry slackness conditions.}$$

Thus while deciding for a variable to enter into the bases at each iteration the complementry slackness conditions must be satisfies .

Step 3:- Apply two phase of simplex method to get optimum solution.

Question :→ 1 Wolef's modified simplex method :→

$$\text{Max } Z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$$

Subject to, $x_1 + x_2 \le 2$

& $x_1, x_2 \ge 0$

Solⁿ :→

Step 1 ;- (a) Firstly, to convert the objective function into maximize form

(b) Write all constraint inequalities with $\le$ sign

{ Since we have two decision variables, so we have two new constraint

$$x_1 + 2x_2 \le 2$$
$$x_1 \cancel{\ge} 0$$
$$x_2 \cancel{\ge} 0$$

(c) Convert the inequalities into the equations by adding slack variables

$$x_1 + 2x_2 + S_1^2 = 2$$
$$-x_1 + r_1^2 = 0$$
$$-x_2 + r_2^2 = 0$$

{ Since it is Quadratic programming so the slack variable are also in square form.

(d) To obtain the Kuhn - tucker condition, we form the langrange's func

$$L(x_1, x_2, S_1, r_1, r_2, \lambda, \mu_1, \mu_2) = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2 - \lambda(x_1 + 2x_2 + S_1^2 - 2) - \mu_1(-x_1 + r_1^2) - \mu_2(-x_2 + r_2^2).$$

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(e) Now, Firstly, find the partial derivaties. The necessary & sufficient conditions are

$$\frac{\partial L}{\partial x_1} = 4 - 2x_2 - d + u_1 - 4x_1 = 0$$

$$\frac{\partial L}{\partial x_2} = 6 - 2x_1 - 2d_1 + u_2 - 4x_2 = 0$$

$$\frac{\partial L}{\partial d_1} = -x_1 - 2x_2 - s_1^2 + 2 = 0$$

$$\frac{\partial L}{\partial u_1} = u_1 x_1 = 0$$

& $\frac{\partial L}{\partial u_2} = u_2 x_2 = 0$

$$\frac{\partial L}{\partial s_1} = d_1 s_1^2 = 0$$

Assume, $s_1^2 = s_1$.

The complementry slackness conditions ($u_1 x_1 = 0$, $u_2 x_2 = 0$ & $d_1 s_1 = 0$) are helpful to find the leaving & entering vector.

Step2:- Construct the modified LPP.

Introducing the artifical variables A_1 & A_2 the modified L.P.P becomes

$$\text{Max } Z = -A_1 - A_2$$

$4x_1 + 2x_2 + d_1 - u_1 + A_1 = 4$ {it is from $\frac{\partial L}{\partial x_1} = 0$}

$2x_1 + 2d_1 - u_2 + 4x_2 + A_2 = 6$ {from $\frac{\partial L}{\partial x_2} = 0$}

$x_1 + 2x_2 + s_1 = 2$ {Here, we dont need any artifical variable b/c we have identity as s_1}

& Complomentry slackness conditions are.

$u_1 x_1 = 0$; $u_2 x_2 = 0$ & $d_1 s_1 = 0$

Matrix representation of the above problem is

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & d_1 & u_1 & u_2 & A_1 & A_2 & S_1 \end{array} \\ \begin{bmatrix} 4 & 2 & 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 4 & 2 & 0 & -1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ d_1 \\ u_1 \\ u_2 \\ A_1 \\ A_2 \\ S_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}$$

Step 3:→ Construct intial table of Phase-I.

Basic var	CB	Cj / XB	0 x_1	0 x_2	0 d_1	0 u_1	0 u_2	-1 A_1	-1 A_2	0 S_1	Min Ratio
A_1	-1	4	$\boxed{4}$	2	1	-1	0	1	0	0	1 →
A_2	-1	6	2	4	2	0	-1	0	1	0	3
S_1	0	2	1	2	0	0	0	0	0	1	2
$Z = C_B X_B = -10$		Δ_j	-6 ↑	-6	-3	1	1	0	0	0	

Remark:→ * With the exceptional condition of complementary slackness. we need to modified, the slackness conditions.. Thus while to introduce S_i we must first ensure that

(1) either d_i is the solⁿ (in basic variable column)

(2) d_i will be remove when S_i enter. (in basic variable column)

* Artifical variable should be removed first from basic variable column

Basic Variable	C_B	C_j / X_B	0 / x_1	0 / x_2	0 / d_1	0 / u_1	0 / u_2	-1 / A_1	-1 / A_2	0 / S_1	Min Ratio
x_1	0	1	1	1/2	1/4	-1/4	0	1/4	0	0	2
A_2	-1	4	0	3	3/2	-1/2	-1	-1/2	1	0	4/3
S_1	0	1	0	$\boxed{3/2}$	-1/4	1/4	0	-1/4	0	1	2/3 →
$z=-4$	-	Δ_j	0	-3↑	-3/2	-1/2	1	3/2	0	0	

Remark

Here d_1 cannot be entring vector in basic soln since S_1 is in basic soln.

Similarly for u_1. due to the slackness condition.

Basic Variable	C_B	C_j / X_B	0 / x_1	0 / x_2	0 / d_1	0 / u_1	0 / u_2	-1 / A_1	-1 / A_2	0 / S_1	Min Ratio
x_1	0	2/3	1	0	1/3	-1/3	0	1/3	0	-1/3	2
A_2	-1	2	0	0	$\boxed{2}$	0	-1	0	1	-2	1 →
x_2	0	2/3	0	1	-1/6	1/6	0	-1/6	0	2/3	—
$Z=-2$	-	Δ_j	0	0	-2↑	0	1	1	1	2	

{ Here d_1 can enter in basic variable column }.

Basic Variable	C_B	C_j / X_B	0 / x_1	0 / x_2	0 / d_1	0 / u_1	0 / u_2	-1 / A_1	-1 / A_2	0 / S_1	Min Ratio
x_1	0	1/3	1	0	0	-1/3	1/6	1/3	-1/6	0	
d_1	0	1	0	0	1	0	-1/2	0	1/2	-1	
x_2	0	5/6	0	1	0	1/6	-1/2	-1/6	1/2	1/2	
$Z=0$	-	Δ_j	0	0	0	0	0	1	1	0	

Since both A_1 & A_2 are out of the basic soln.

The optimum soln is $x_1 = 1/3$, $x_2 = 5/6$, Max $Z = 5/6$.

Q2:- Apply Wolfe's method to solve it.

$$\text{Max } Z = 2x_1 + x_2 - x_1^2 \qquad x_1 = \frac{2}{3},\ x_2 = \frac{14}{9},\ Z = \frac{22}{9} \text{ Ans}$$

s. to

$$2x_1 + 3x_2 \le 6$$
$$2x_1 + x_2 \le 4$$
$$\& \quad x_1, x_2 \ge 0$$

$$\text{Max } Z = 2x_1 + x_2 - x_1^2$$
$$2x_1 + 3x_2 + s_1^2 = 6$$
$$2x_1 + x_2 + s_2^2 = 4$$
$$-x_1 + r_1^2 = 0$$
$$-x_2 + r_2^2 = 0$$

$$L(x, s, r, d, u) = (2x_1 + x_2 - x_1^2) - d_1(2x_1 + 3x_2 + s_1^2 - 6) -$$
$$d_2(2x_1 + x_2 + s_2^2 - 4) - u_1(-x_1 + r_1^2) - u_2(-x_2 + r_2^2)$$

$$\frac{\partial L}{\partial x_1} = 2 - 2x_1 - 2d_1 - 2d_2 + u_1 = 0$$

$$\frac{\partial L}{\partial x_2} = 1 - 3d_1 - d_2 + u_2 = 0$$

$$\frac{\partial L}{\partial d_1} = -(2x_1 + 3x_2 + s_1^2 - 6) = 0$$

$$\frac{\partial L}{\partial d_2} = 2x_1 + x_2 + s_2^2 = 4$$

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$$\left.\begin{array}{l} -2d_1 s_1 = 0 \\ -2d_2 s_2 = 0 \\ -2\mu_1 s_{11} = 0 \;,\; -2\mu s_{12} = 0 \end{array}\right\}$$ slackness variable

The construct modified linear programming problem.

$$2x_1 + 2d_1 + 2d_2 - \mu_1 + A_1 = 2$$

$$3d_1 + d_2 - \mu_2 + A_2 = 1$$

$$2x_1 + 3x_2 + s_1 = 6$$

$$2x_1 + x_2 + s_2 = 4$$

B.V.	CB	c_j / X_B	0 / x_1	0 / x_2	0 / d_1	0 / d_2	0 / μ_1	0 / μ_2	-1 / A_1	-1 / A_2	0 / S_1	0 / S_2	Min Ratio
A_1	-1	2	$\boxed{2}$	0	2	2	-1	0	1	0	0	0	1 →
A_2	-1	1	0	0	3	1	0	-1	0	1	0	0	
S_1	0	6	2	3	0	0	0	0	0	0	1	0	3
S_2	0	4	2	1	0	0	0	0	0	0	0	1	2
$Z = -3$		Δ_j	2↑	0	5	3	-1	-1	0	0	0	0	
x_1	0	1	1	0	1	1	-1/2	0	1/2	0	0	0	—
A_2	-1	1	0	0	3	1	0	-1	0	1	0	0	:3
S_1	0	4	0	$\boxed{3}$	-2	-2	1	0	-1	0	1	0	1.33 →
S_2	0	2	0	1	-2	-2	1	0	-1	0	0	1	2
$Z = -1$		Δ_j	0	0↑	3	1	0	-1	-1	0	0	0	
x_1	0	1	1	0	1	1	-1/2	0	1/2	0	0	0	1
A_2	-1	1	0	0	$\boxed{3}$	1	0	-1	0	1	0	0	1/3 →
x_2	0	4/3	0	1	-2/3	-2/3	1/3	0	-1/3	0	1/3	0	—
S_2	0	2/3	0	0	-4/3	-4/3	2/3	0	-2/3	0	-1/3	1	—
$Z = -1$		Δ_j	0	0	3↑	1	0	-1	-1	0	0	0	
x_1	0	2/3	1	0	0	2/3	-1/2	1/3	1/2	-1/3	0	0	
d_1	0	1/3	0	0	1	1/3	0	-1/3	0	1/3	0	0	
x_2	0	14/9	0	1	0	-4/9	1/3	-2/9	-1/3	2/9	1/3	0	
S_2	0	10/9	0	0	0	-8/9	-2/9	-4/9	-2/3	4/9	-1/3	1	
ZZ 0		Δ_j	0	0	0	0	0	0	-1	-1	0	0	

Hence The optimum Soln is

$$x_1 = 2/3$$
$$x_2 = 14/9$$
$$z = 22/9.$$

~~Z = 22/9~~ **Ans**

Soln:- Max $Z = 2x_1 + 3x_2 - x_1^2$

$x_1 + 2x_2 + s_1^2 = 4$

$-x_1 + r_1^2 = 0$

$-x_2 + r_2^2 = 0$

The Langrange's function is

$L(x_1, x_2, s_1, r_1, r_2, d_1, \mu_1, \mu_2) = (2x_1 + 3x_2 - x_1^2) - d_1(x_1 + 2x_2 + s_1^2 - 4)$
$- \mu_1(-x_1 + r_1^2) - \mu_2(-x_2 + r_2^2)$

Kuhn-Tucker conditions as follows-

$\frac{\partial L}{\partial x_1} = 2 - 2x_1 - d_1 + \mu_1 = 0$

$\frac{\partial L}{\partial x_2} = 3 - 2d_1 + \mu_2 = 0$

$\frac{\partial L}{\partial d_1} = x_1 + 2x_2 + s_1 = 4$

$x_1 = r_1$

$x_2 = r_2$

To construct the modified L.P.P is -

Max $Z = 2x_1 + 3x_2 - x_1^2$

~~$x_1 + 2x_2 + s_1 = 4$~~ $2x_1 + d_1 - \mu_1 = 2$

$2d_1 - \mu_2 = 3$

$x_1 + 2x_2 + s_1 = 4$

To construct initial table of phase I.

B.V.	C_B	X_B	x_1	x_2	d_1	μ_1	μ_2	A_1	A_2	S_1	Min Ratio
	C_j		0	0	0	0	0	-1	-1	0	
A_1	-1	2	$\boxed{2}$	0	1	-1	0	1	0	0	1 →
A_2	-1	3	0	0	2	0	-1	0	1	0	-
S_1	0	4	1	2	0	0	0	0	0	1	4
$Z = 0$		Δ_j	2↑	0	1	-1	-1	0	0	0	
x_1	0	1	1	0	1/2	-1/2	0	1/2	0	0	
A_2	-1	3	0	0	2	0	-1	0	1	0	
S_1	0	3	0	$\boxed{2}$	-1/2	1/2	0	-1/2	0	1	
$Z = -3$		Δ_j	0	0↑	2	0	-1	-1	0	0	

CBV	CB	C_j / X_B	0 x_1	0 x_2	0 d_1	0 u_1	0 u_2	-1 A_1	-1 A_2	0 S_1	Min Ratio
x_1	0	1	1	0	1/2	-1/2	0	1/2	0	0	2
A_2	-1	3	0	0	$\boxed{2}$	0	-1	0	1	0	3/2 →
x_2	0	3/2	0	1	-1/4	1/4	0	-1/4	0	1/2	—
$Z = -3$		Δ_j	0	0	2 ↑	0	-1	-1	0	0	
x_1	0	1/4	1	0	0	-1/2	1/4	1/2	-1/4	0	
d_1	0	3/2	0	0	1	0	-1/2	0	1/2	0	
x_2	0	15/8	0	1	0	1/4	-1/8	-1/4	1/8	1/2	
$Z = 0$		Δ_j	0	0	0	0	0	0	0	0	

Since all $\Delta_j \le 0$. Hence the optimal solⁿ is

$x_1 = 1/4$, $x_2 = \frac{15}{8}$, $\text{Max } Z = \frac{97}{16}$ Ans

Method-2

Beale's Method

Beale's Method :- Another approach to solve Q.P.P. has be suggested by Beale's.

This method involves partitioning of variables into basic or non-basic variable. At each iteration the objective func is express in terms of non-basic variable only. let the Q.P.P be of the form.

$$\text{Max } f(x) = cx + \frac{1}{2} x^T Q x$$

s. to $\quad AX = b$

$\quad\quad X \geq 0$

where $X = (x_1, x_2 \;---\; x_{m+n})'$, $C = 1 \times m$

and A is $m \times (n+m)$ and Q is Symmetric matrix.

The Beale's iteration procedure can be out line in following steps.

Step 1:- first express the given Q.P.P with linear constraint in the following form by introducing slack & surplus variable.

Step 2: Select arbitrary m variables as basic & remaining n is non basic. with this partitioning, constraint eqn $AX = b$ can be written as:

$$(B, R)\begin{bmatrix} x_B \\ X_{NB} \end{bmatrix} = b \quad \text{——(i)}$$

or $\quad BX_B + R\, X_{NB} = b$

where X_B - denote basic variable

$\quad\quad X_{NB}$ - denotes the non basic variable.

Also matrix A is partition of sub matrix B & R corresponding to X_B & X_{NB} respectively. According to partitioning eqn (i) can be written as

$$X_B = B^{-1}(b - R x_{NB})$$

Express the basic variable X_B in terms of non basic variable X_{NB} using the given an additional constraint eqn.

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Step 3:- Express the objective func $f(x)$ in terms of $\cancel{x_B}$ & x_{NB}. Thus we observed that by increasing value of any non-basic variable the value of objective func can be improved. It is important to note that the constraint in new problem become

$$B^{-1} R\, x_{NB} = B^{-1} b$$

Thus any component of x_{NB} can increase only until $\frac{\partial f}{\partial x_{NB}}$ becomes zero or one or more components of x_B are reduce to zero.

Step 4:- We now have possibility of having more than m non-zero variable at any step of iteration. This stage comes when the new point is generated at some steps occurs, where $\frac{\partial f}{\partial x_{NB}}$ becomes zero.

Geometrically it means that we no longer have a basic soln with respect to the original constraint set. When this happens we define a new variable $S_i = \frac{\partial f}{\partial x_{NB}}$ & the new constraint $S_i = 0$

Step 5:- We now have $m+1$ non-basic variable & $m+1$ constraint which is a basic soln to the extended set of constraints. We repeat the outline procedure until no further improvement in the objective func may be obtained by increasing one of the non basic variable.

Beale's Method :-

Q. Min $z = -4x_1 + x_1^2 - 2x_1x_2 + 2x_2^2$

Subject to, $2x_1 + x_2 \geq 6$

$x_1 - 4x_2 \geq 0$

& $x_1; x_2 \geq 0$

Sol^n:-

Step 1:- Convert the minimization problem into maximization & adding surplus variables s_1 and s_2

$$\text{Max } z = 4x_1 - x_1^2 + 2x_1x_2 - 2x_2^2$$

$$2x_1 + x_2 - s_1 = 6$$

$$x_1 - 4x_2 - s_2 = 0$$

Step 2:- Selecting arbitrary m variables as basic variables & so that remaining n-m variables become non-basic variable

Here, Making s_1 & s_2 basic variable in the initial sol^n and expressing these in terms of non-basic variables x_1 & x_2

Here $X_B = s_1$ & s_2 & $X_{NB} = x_1$ & x_2

$$s_1 = -6 + 2x_1 + x_2$$

$$s_2 = x_1 - 4x_2$$

$$f(x) = z = 4x_1 - x_1^2 + 2x_1x_2 - 2x_2^2$$

Step 3:-

Examine the partial derivate of $f(x)$ formulate with respect to non-basic variable & put the non-basic variable equal to zero

$$\left.\frac{\partial z}{\partial x_1}\right|_{\substack{x_1=0\\x_2=0}} = 4 - 2x_1 - 2x_2 \Big|_{\substack{x_1=0\\x_2=0}} = 4$$

$$\left.\frac{\partial z}{\partial x_2}\right|_{\substack{x_1=0\\x_2=0}} = -2x_1 + 2x_1 - 4x_2' = 0 \Big|_{\substack{x_1=0\\x_2=0}}$$

Step 4 :- If $\left[\frac{\partial f(z)}{\partial x_{NB}}\right]_{x_{NB}=0} > 0$ for at least one.

Chose the most positive one & the corresponding non-basic variable will enter in the basis

Here $\left.\frac{\partial z}{\partial x_1}\right|_{\substack{x_1=0\\x_2=0}} = 4 > 0$

Clearly x_1 will enter in the basic. Compute the minimum ratio. $\text{Min}\left\{\frac{\alpha}{\text{coeff of } x_1 \text{ in the constraint}}, \frac{\gamma}{\text{coff of } x_1 \text{ in the corr constraint}}\right\}$

$\text{Min}\left\{\frac{-6}{|2|}, \frac{0}{\text{coff}(x_1)}\right\}$

If the minimum ratio occurs $\underline{\text{d}_{rk}}$ the corresponding basic variable leave the basis basic variable leave the basis.

i.e The variable s_1 is eligible to leave the basis.

Step 5 :

Now Expressing the new basic variable x_1, s_2 fuis terms of new non basic variables x_2 & s_1

$$x_1 = 3 - \frac{1}{2}x_2 + 1/2\, s_1$$

$$s_2 = 3 - 1/2\, x_2 + 1/2\, s_1 - 4x_2$$

{ but the value of x_1 because it is basic variable now

$$= 3 - \frac{9}{2}x_2 + 1/2\, s_1$$

$$z = 4x_1 - x_1^2 + 2x_1x_2 - 2x_2^2$$ { but the value of x_1 in z }

$$z = 4\left(3 - 1/2\, x_2 + 1/2\, s_1\right) - \left(3 - 1/2\, x_2 + 1/2\, s_1\right)^2 + 2\left(3 - 1/2\, x_2 + 1/2\, s_1\right)x_2 - 2x_2^2$$

$$z = 9 + x_2 - s_1 + \frac{3}{2}x_2 s_1 - \frac{13}{4}x_2^2 - \frac{1}{4}s_1^2$$

Here $X_B = (x_1, s_2)$ & $X_{NB} = (x_2, s_1)$.

Step 6

Again Diff z with respect to non-basic variable & put the non-basic variables equal to zero,

$$\left.\frac{\partial z}{\partial x_2}\right|_{\substack{x_2=0\\ s_2=0}} = 1 + \frac{3}{2}s_1 - \frac{13}{2}x_2\Big|_{\substack{x_2=0\\ s_2=0}} = 1$$

$$\left.\frac{\partial z}{\partial s_1}\right|_{\substack{x_2=0\\ s_1=0}} = -1 + \frac{3}{2}x_2 - \frac{1}{4}s_1\Big|_{\substack{x_2=0\\ s_1=0}} = -1$$

Chose the most positive one & the correspondic non-basic variable will enter in the basis

Here $\frac{\partial z}{\partial x_2} = 1 > 0$

clearly x_2 will enter in the basic. Now, Compute the minimum ratio

$$\text{Min}\left\{\frac{+3}{-1/2}, \frac{3}{-9/2}\right\} = \text{Min}\{6, 6/9\} = 6/9$$

$\{\alpha, \beta\}$

If the minimum ratio r the exit criterion corresponding to a non-basic variable. In this case introduce ~~artificial~~ additional non-basic variable called a free variable

defined by $u = 1/2\ \partial z/\partial x_2$

Step 7

Since the minimum ratio is corresponding to β. We introduce a non-basic free variable u_1, defined by

$$u_1 = 1/2\ \partial z/\partial x_2 = 1/2 + 3/4\, s_1 - 13/4\, x_2$$

Expressing basic variable & z in terms of X_{NB}

$X_B = (x_1, x_2, s_2)$ $\quad X_{NB} = (s_1, u_1)$

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$$x_2 = \frac{2}{13} + \frac{3}{13}s_1 - \frac{4}{13}u_1 \quad \text{—— (A)}$$

$$x_1 = \frac{30}{13} - \frac{3}{26}s_1 + \frac{2}{13}u_1 \quad \text{—— (B)}$$

$$s_2 = \frac{30}{13} - \frac{27}{26}s_1 + \frac{10}{13}u_1 \quad \text{—— (C)}$$

$$\& \; z = 9 + \frac{1}{13}[2+3s_2-4u_2] - s_2 - \frac{3}{26}[2+3s_1-4u_1]s_1$$
$$-1/52\,[2+3s_2-u_1)^2 - 1/4\, s_1^2$$

Again

$$\left.\frac{\partial z}{\partial s_1}\right|_{\substack{s_1=0\\u_1=0}} = -9/13$$

$$\left.\frac{\partial z}{\partial u_1}\right|_{\substack{s_1=0\\u_1=0}} = 0$$

Here $\frac{\partial z}{\partial s_1} < 0$ & $\frac{\partial z}{\partial u_1} = 0$

So the optimum value of z is obtained by setting $u_1=0$ & $s_1=0$ in the current value of objective func

$$z^* = 9 + \frac{2}{13} - \frac{2}{52} = \frac{474}{52}$$

Hence the optimal solⁿ of given problem is put the non-basic variable equal zero in (A), (B) & C

$$\left.\begin{aligned} x_1 &= 30/13, \; x_2 = 2/13 \\ \& \; \min &= 9.115 \end{aligned}\right\} \underline{\text{Ans}}$$

$z = 100$

Q2:- Max $z = 10x_1 + 25x_2 - 10x_1^2 - x_2^2 - 4x_1x_2$

$x_1 + 2x_2 + x_3 = 10$

$x_1 + x_2 + x_4 = 9$

$x_1, x_2, x_3, x_4 \geq 0$

same as last Question.

Selecting x_1 & x_2 arbitrarily to be the basic variables we obtain $x_1 = 8 + x_3 - 2x_4$, $x_2 = 1 - x_3 + x_4$ where

$X_B = (x_1, x_2)$, $X_{NB} = (x_3, x_4)$

Step 2:- Now expressing z_x in terms of (x_3, x_4) gives.

$f(x_3, x_4) = 10(8 + x_3 - 2x_4) + 25(1 - x_3 + x_4) + 10(8 + x_3 - 2x_4)^2 - (1 - x_3 + x_4)^2 - 4(8 + x_3 - 2x_4)(1 - x_3 + x_4)$

$$\frac{\partial f(x_{NB})}{\partial x_3} = 10 - 25 - 20(8 + x_3 - 2x_4) + 2(1 - x_3 + x_4) - 4(1 - x_3 + x_4) + 4(8 + x_3 - 2x_4)$$

$$\left(\frac{\partial f}{\partial x_3}\right)_{x_3 = x_4 = 0} = -145$$

This indicates that objective func will decrease if x_3 is increased. This happens contrary to our desire to increase the objective function. The partial derivative with respect to x_4 will give us a more suitable alternative.

$$\left(\frac{\partial f}{\partial x_4}\right) = -20 + 25 + 40(8 + x_3 - 2x_4) - 2(1 - x_3 + x_4) + 8(1 - x_3 + x_4) - 4(8 + x_3 - 2x_4)$$

$$\left(\frac{\partial f}{\partial x_4}\right)_{x_3 = x_4 = 0} = 299$$

Step 3:- If x_4 is increased to a value greater than 4, x_1 will becomes negative since $x_1 = 8 + x_3 - 2x_4$ & $x_3 = 0$ The partial derivates becomes zero at $x_4 = \frac{299}{66}$

Taking minimum of $\left(4, \frac{299}{66}\right)$, we find $x_4 = 4$, and new basic variables are x_4 & x_2. We now start to with new iteration.

Second iteration:- Step 4:- We start with solving for x_2 & x_4 in terms of x_1 & x_3. Thus.

$$x_2 = 5 - \frac{1}{2}(x_1 + x_3), \quad x_4 = 4 + \frac{1}{2}(x_3 - x_1)$$

$$x_B = (x_2, x_4), \quad x_{NB} = (x_1, x_3)$$

expressing Zx in terms of (x_1, x_3) gives.

$$f(x_1, x_3) = 10x_1 + 25\left[5 - 1/2(x_1 + x_3)\right] - 10x_1^2 - \left[5 - 1/2(x_1 + x_3)\right]^2$$
$$- 4x_1\left[5 - 1/2(x_1 + x_3)\right]$$

$$\left(\frac{\partial f}{\partial x_1}\right)_{x_1 = x_2 = 0} = \frac{-15}{2}$$

Since both the partial derivatives are -ive, hence neither x_1 nor x_3 non-basic variables can be introduced to increase Zx and thus the optimal soln has been obtain

The optimal soln is given by.

$x_1 = x_3 = 0, \; x_2 = 5, \; x_4 = 4$ Ans

Q6 $\quad \text{Max } z = 2x_1 + x_2 - x_1^2$

$$2x_1 + 3x_2 \le 6$$
$$2x_1 + x_2 \le 4$$
$$x_1, x_2 \ge 0$$
$$2x_1 + 3x_2 + x_3 = 6$$
$$2x_1 + x_2 + x_4 = 4$$

Selecting x_1 & x_2 arbitrarily to the basic variables

we obtain $\quad 2x_2 - x_3 - x_4 = 2$ $\qquad 2x_2 + x_3 - x_4 = 2$

$$x_1 = \frac{1}{2}(2 + x_3 + x_4) \qquad x_2 = \frac{1}{2}(2 + x_3 + x_4)$$

& $x_2 =$

$$-4x_1 + x_3 - 3x_4 = -6$$
$$x_1 = \frac{1}{4}(6 + 3x_4 - x_3)$$

Putting the value of x_1 & x_2 in eqn (i), the func

$$f = \frac{1}{2}(6 + x_3 - 3x_4) + \frac{1}{2}(2 - x_3 + x_4) - \left(\frac{3}{2} + \frac{1}{4}x_3 - \frac{3}{4}x_4\right)^2$$

$$\frac{\partial f}{\partial x_3} = \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} - 2\left[\frac{3}{2} + \frac{1}{4}x_3 - \frac{3}{4}x_4\right]\frac{1}{4}$$

$$= -\frac{3}{4} - \frac{1}{8}x_3 + \frac{3}{8}x_4$$

$$\frac{\partial f}{\partial x_4} = \frac{5}{4} + \frac{3}{8}x_3 - \frac{9}{8}x_4$$

$$\left.\frac{\partial f}{x_3}\right|_{x_3 = x_4 = 0} = -3/4$$

$$\left.\frac{\partial f}{\partial x_4}\right|_{x_3 = x_4 = 0} = 5/4$$

Since x_3 is negative (x_4) gives better improvement to objective func by increased the value of x_4, now,

Since x_1 gives negative value when x_4 increase to grater than 2. and

$$\left.\frac{\partial f}{\partial x_4}\right| = 0 \Rightarrow \frac{5}{4} - \frac{9}{8}x_4 \Rightarrow x_4 = 10/9$$

$\min(x_4) = \min(2, 10/9) = 10/9$ Ans

Iteration-2:- Now, let $x_B = (x_2, x_4)$ & $X_{NB} = (x_1, x_3)$

$x_2 = \frac{1}{3}(6 - x_3 - 2x_1)$, $x_4 = 4 - 2x_1 - x_2$

put the value of x_2 & x_4 in eq (i)

$$f = 2x_1 + \frac{1}{3}(6 - x_3 - 2x_1) - x_1^2$$

$$\frac{\partial f}{x_1} = 2 - \frac{2}{3} - 2x_1 = \frac{4}{3} - 2x_1$$

$$\left.\frac{\partial f}{\partial x_1}\right|_{x_1 = x_3 = 0} = 4/3$$

$$\frac{\partial f}{\partial x_3} = -1/3$$ (Which is -ive.

Since x_1 is gives better improvement to objective func by increased the value of x_1, now,

$\left.\frac{\partial f}{\partial x}\right|_{x_3=0} = \frac{4}{3} - 2x_1 = 0 \Rightarrow x_1 = 2/3$

$\min(x_1) = \min(3, 2/3) = 2/3$

<u>Iteration III</u>:- Now let $x_B = (x_1, x_4)$ & $x_{NB} = (x_2, x_3)$

$x_1 = 1/2(6 - 3x_2 - x_3)$, $x_4 = 4 - 2x_1 - x_2$

$f = (6 - 3x_2 - x_3) + x_2 - \frac{1}{4}(6 - 3x_2 - x_3)^2$

$\frac{\partial f}{\partial x_2} = 7 - \frac{9}{2}x_2 + \frac{3}{2}x_3 \qquad \left.\frac{\partial f}{\partial x_2}\right|_{x_2 = x_3 = 0} = 7$

$\left.\frac{\partial f}{\partial x_3}\right| = 2 - \frac{3}{2}x_3 - \frac{1}{2}x_3 \qquad \left.\frac{\partial f}{\partial x_3}\right|_{x_2 = x_3 = 0} = 2$

$\left.\frac{\partial f}{\partial x_2}\right|_{x_3=0} = 7 - \frac{9}{2}x_2 = 0 \Rightarrow x_2 = 14/9$

$\min(x_2) = 14/9$

Hence the Optimal soln is $x_1 = 2/3$, $x_2 = 14/9$, $z = 22/9$ Ans