Chapter - 5

Eigen Values, Eigen Vectors, Orthogonal Matrix and Orthogonal Vectors

1. Characteristic Matrix, Characteristic Equation and Characteristic Roots of a Matrix –

Let A be an $n \times n$ matrix and λ be a variable. The matrix $(A - \lambda I)$ is called the characteristic matrix of A and its determinant viz. $|A - \lambda I|$ is known as characteristic function which is a polynomial of degree n in λ. The characteristic function $|A - \lambda I|$ equated to zero gives the characteristic equation of A. The roots of the characteristic equation $|A - \lambda I| = 0$ are called the characteristic roots or latent roots or eigen values of matrix A. The set of the eigen values of A is called the spectrum of A.

If $\lambda_1, \lambda_2, \cdots \lambda_n$ are the characteristic roots of matrix A then

$$|A - \lambda I| = 0 \quad \forall \quad \lambda = \lambda_1, \lambda_2, \cdots, \lambda_n$$

and the matrix $(A - \lambda I)$ is singular. Therefore there exists a non-zero vector X such that

$$(A - \lambda I) X = 0 \quad \text{or} \quad AX = \lambda X$$

Thus 'Any non-zero vector X is said to be a characteristic vector of a matrix A if there exists a number λ such that $AX = \lambda X$'. Also then λ is said to be a characteristic root of the matrix A.

corresponding to the characteristic vector X and vice-versa.

Theorem-1 Every square matrix A satisfies its characteristic equation (Cayley Hamilton th.)

Proof— The theorem states that if
$$|A-\lambda I| = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n = 0$$
be the characteristic equation of a n-rowed square matrix A, then
$$a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n = 0$$
To prove the theorem, we know that the elements of $(A-\lambda I)$ are at most of the first degree in λ and elements of adj $(A-\lambda I)$ are at most of degree $(n-1)$ in λ. Therefore adj $(A-\lambda I)$ may be written as a matrix polynomial in λ. Suppose
$$\operatorname{adj}(A-\lambda I) = B_0 + B_1\lambda + B_2\lambda^2 + \cdots + B_{n-1}\lambda^{n-1}$$
where $B_0, B_1, \ldots, B_{n-1}$ are $n\times n$ matrices.

Now using the relation $A \operatorname{adj}(A) = |A| I$, we have
$$(A-\lambda I)\operatorname{adj}(A-\lambda I) = |A-\lambda I|\, I$$
or $$(A-\lambda I)(B_0 + B_1\lambda + \cdots + B_{n-1}\lambda^{n-1}) = (a_0 + a_1\lambda + \cdots + a_n\lambda^n) I$$
$$= a_0 I + a_1 I\lambda + a_2 I\lambda^2 + \cdots + a_n I\lambda^n$$
Comparing both sides, we have
$$AB_0 = a_0 I$$
$$AB_1 - B_0 = a_1 I$$
$$AB_2 - B_1 = a_2 I$$
$$\cdots\cdots\cdots$$
$$-B_{n-1} = a_n I$$

Pre-multiplying these successively with

$I, A, A^2, \ldots, A^n$ and adding we get

$$(Ia_0 + Aa_1 + A^2 a_2 + \cdots + A^n a_n) I = 0$$

or $a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n = 0$ Proved

Corollary — From the above theorem, we have

$$a_0 I = -a_1 A - a_2 A^2 - \cdots - a_n A^n$$

or $I = -\frac{a_1}{a_0} A - \frac{a_2}{a_0} A^2 - \cdots - \frac{a_n}{a_0} A^n$

Pre-multiplying with A^{-1}, we obtain

$$A^{-1} = -\frac{a_1}{a_0} I - \frac{a_2}{a_0} A - \cdots - \frac{a_n}{a_0} A^{n-1}$$

which is the required expression for A^{-1}

Exercise - 1 Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

and verify that it is satisfied by A and hence obtain A^{-1}.

Solution — We have the characteristic equation

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$$

It is now to be verified that

$$-A^3 + 6A^2 - 9A + 4I = 0$$

Now I, A, A^2 and A^3 are the matrices

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$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}, \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}, \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

respectively and the verification may easily be computed.

Again

$$A^{-1} = \frac{1}{4} A^2 - \frac{6}{4} A + \frac{9}{4} I = \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$ after simplification

Theorem-2 If X is a characteristic vector of a matrix A, then X can't correspond to more than one characteristic values of A.

Proof— Let X be a characteristic vector of a matrix A corresponding to two characteristic values λ_1 and λ_2. Then

$$AX = \lambda_1 X \quad \text{and} \quad AX = \lambda_2 X$$

Therefore $\lambda_1 X = \lambda_2 X \Rightarrow (\lambda_1 - \lambda_2) X = 0$

$$\Rightarrow (\lambda_1 - \lambda_2) = 0 \quad \because X \neq 0$$
$$\Rightarrow \lambda_1 = \lambda_2$$

Theorem-3 If λ is a characteristic root of a non-singular matrix A, then λ^{-1} is a characteristic root of A^{-1}.

Proof— Given λ is a characteristic root of a non-singular matrix A. $\therefore |A| \neq 0$ and

$$|A - \lambda I| = 0 \Rightarrow |A - \lambda A A^{-1}| = 0$$
$$\Rightarrow |A||I - \lambda A^{-1}| = 0 \Rightarrow |\lambda A^{-1} - I| = 0 \quad \because |A| \neq 0$$
$$\Rightarrow |A^{-1} - \lambda^{-1} I| = 0 \Rightarrow 1/\lambda \text{ is a ch root of } A^{-1}$$

Proved

<u>Theorem-4</u> Show that the two matrices A and $C^{-1}AC$ have the same characteristic roots

<u>Proof</u>— Let $B = C^{-1}AC$ then

$$\begin{aligned} B - \lambda I &= C^{-1}AC - \lambda I \\ &= C^{-1}AC - C^{-1}C\lambda I \\ &= C^{-1}AC - C^{-1}\lambda I C \\ &= C^{-1}(A - \lambda I)C \end{aligned}$$

$$\begin{aligned} \therefore\ |B - \lambda I| &= |C^{-1}|\,|A - \lambda I|\,|C| \\ &= |A - \lambda I|\,|C^{-1}|\,|C| = |A - \lambda I|\,|C^{-1}C| \\ &= |A - \lambda I| \end{aligned}$$

Thus the two matrices A and B have the same characteristic functions and hence the same characteristic equations and the same characteristic roots.

<u>Theorem-5</u> Show that a characteristic vector X corresponding to the characteristic root λ of a matrix A is also a characteristic vector of every matrix $f(A)$ with the corresponding root ~~...~~ $f(\lambda)$. $f(\cdot)$ being any scalar polynomial. In general, show that if $g(x) = \dfrac{f_1(x)}{f_2(x)}$; $|f_2(x)| \neq 0$

then $g(\lambda)$ is a characteristic root of $g(A) = \dfrac{f_1(A)}{f_2(A)}$

<u>Proof</u>— λ is a characteristic root of A

$\therefore \quad AX = \lambda X$

Suppose $f(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n$

then $f(A)X = (a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n)X$

$$= (a_0 I X + a_1 A X + a_2 A^2 X + \cdots + a_n A^n X)$$
$$= a_0 X + a_1 \lambda X + a_2 \lambda^2 X + \cdots + a_n \lambda^n X$$
$$= (a_0 + a_1 \lambda + a_2 \lambda^2 + \cdots + a_n \lambda^n) X = f(\lambda) X$$

$\because AX = \lambda X$
$\Rightarrow AAX = \lambda AX$
$\Rightarrow A^2 X = \lambda^2 X$
Similarly others

Hence $f(\lambda)$ is the characteristic root of the matrix $f(A)$ and X is the corresponding characteristic vector.

In general

$$g(A) X = \frac{f_1(A)}{f_2(A)} X = f_1(A) \{ f_2(A) \}^{-1} X$$
$$= f_1(A) \{ f_2(\lambda) \}^{-1} X$$
$$= \{ f_2(\lambda) \}^{-1} f_1(A) X$$
$$= \{ f_2(\lambda) \}^{-1} f_1(\lambda) X = g(\lambda) X$$

Thus X is also a characteristic vector of $g(A)$ with corresponding root $g(\lambda)$ ___

<u>Theorem-6</u> If A and B are two square matrices then the matrix AB and BA have the same characteristic roots.

<u>Proof</u>- Let λ be the ch. root of matrix AB. Therefore

$$AB - \lambda I = B^{-1} B A B - \lambda I$$
$$= B^{-1} B A B - B^{-1} B \lambda I = B^{-1} B A B - B^{-1} \lambda I B$$
$$= B^{-1} (BA - \lambda I) B$$
$$\Rightarrow |AB - \lambda I| = |B^{-1}| \, |BA - \lambda I| \, |B|$$
$$= |BA - \lambda I| \, |B^{-1}| \, |B| = |BA - \lambda I| \, |B^{-1} B|$$
$$= |BA - \lambda I|$$

Thus AB and BA have the same characteristic functions and hence the same characteristic equation and the same characteristic roots ___

Theorem-7 The characteristic roots of a Hermitian matrix are all real

Proof- Let λ be a characteristic root of a Hermitian matrix A so that there exist a vector $X \neq 0$ such that

$$AX = \lambda X$$

$$\Rightarrow X^{\Theta} A X = \lambda X^{\Theta} X$$

$$= \lambda X^{\Theta} I X$$

$$\Rightarrow \lambda = \frac{X^{\Theta} A X}{X^{\Theta} I X}$$

Now $\overline{X^{\Theta} A X} = \overline{(X^{\Theta} A X)'}$ $\because X^{\Theta} A X$ is scalar

$$= (X^{\Theta} A X)^{\Theta}$$

$$= X^{\Theta} A^{\Theta} X$$

$$= X^{\Theta} A X \quad \because A^{\Theta} = A$$

$\Rightarrow X^{\Theta} A X$ is real.

Similarly we can show

$$\overline{X^{\Theta} I X} = X^{\Theta} I X$$

As $X^{\Theta} A X$ and $X^{\Theta} I X$ both are real, therefore λ is real.

Theorem-8 The characteristic roots of a Skew-Hermitian matrix is either zero or a pure imaginary number.

Proof- Let λ be the characteristic root of a Skew-Hermitian matrix A so that there exist a vector $X \neq 0$ such that

$$AX = \lambda X$$

$$\text{or } (iA)X = (i\lambda)X \quad \text{———} \quad (1)$$

For skew-hermitian matrix, $-A^{\Theta} = A$

Now $(iA)^{\Theta} = -iA^{\Theta} = iA \Rightarrow iA$ is hermitian

$\cancel{-A^{\theta} = A}$

or $\cancel{-(\bar{A})' = A}$

or $\cancel{-i(\bar{A})' = iA}$

or $\cancel{-i\bar{A} = (iA)'}$

or $\cancel{(-i\bar{A}) = (\overline{iA})'}$

or $\cancel{iA = (iA)^{\theta}}$

or $\cancel{iA^{\theta} = iA}$

or $\cancel{(-iA^{\theta})^{\theta} = (iA)^{\theta}}$

or $\cancel{iA = (iA)^{\theta}}$

As iA is hermitian and (1) shows that the characteristic root of iA is $i\lambda$, therefore according to theorem-7 $i\lambda$ must be real. It will be possible only when $\lambda = 0$ or a pure imaginary number.

Corr— A characteristic root of a real skew symmetric matrix is either zero or pure imaginary because a real skew symmetric matrix is always skew-hermitian.

Theorem-9 The characteristic roots of a real symmetric matrix are all real.

Proof— If the elements of a Hermitian matrix A are all real then A is a real symmetric matrix. Thus a real symmetric matrix is hermitian and therefore the result follows by theorem-7.

Theorem-10 Any system of characteristic vectors $X_1, X_2 \cdots X_k$ corresponding respectively to a system of distinct characteristic roots $\lambda_1, \lambda_2, \cdots, \lambda_k$ of a matrix is linearly indep.

Proof— We have $AX_1 = \lambda_1 X_1$, $AX_2 = \lambda_2 X_2$, ..., $AX_k = \lambda_k X_k$

$\Rightarrow (A-\lambda_1 I)X_1=0,\ (A-\lambda_2 I)X_2=0,\ \ldots,\ (A-\lambda_k I)X_k=0$ (1)

Consider any relation

$$a_1X_1+a_2X_2+\cdots+a_kX_k=0 \qquad (2)$$

Pre-multiplying (2) by the matrix

$$L_1=(A-\lambda_2 I)(A-\lambda_3 I)\cdots(A-\lambda_k I) \qquad (3)$$

we get

$$L_1(a_1X_1)+L_1(a_2X_2)+\cdots+L_1(a_kX_k)=0 \qquad (4)$$

Now by (1) and (3)

$$L_1X_2=0,\ L_1X_3=0,\ \ldots,\ L_1X_k=0 \qquad (5)$$

and $L_1X_1=(A-\lambda_2 I)(A-\lambda_3 I)\cdots(A-\lambda_k I)X_1$

$=(A-\lambda_2 I)(A-\lambda_3 I)\cdots(AX_1-\lambda_k X_1)$

$=(A-\lambda_2 I)(A-\lambda_3 I)\cdots(\lambda_1X_1-\lambda_k X_1)$ $\quad \because AX_1=\lambda_1X_1$

$=(A-\lambda_2 I)(A-\lambda_3 I)\cdots(\lambda_1-\lambda_k)X_1$

$\vdots$

$=(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\cdots(\lambda_1-\lambda_k)X_1$ (6)

Further, using (5) in (4) we get

$L_1(a_1X_1)=0$

$\Rightarrow a_1L_1X_1=0$

$\Rightarrow a_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\cdots(\lambda_1-\lambda_k)X_1=0$, using (6)

Now, as λ_1 is distinct from each $\lambda_2, \lambda_3, \ldots, \lambda_k$ and $X_1\neq 0$,

therefore $a_1=0$

In general, pre-multiplying (2) by

$$L_i=(A-\lambda_1 I)\cdots(A-\lambda_{i-1} I)(A-\lambda_{i+1} I)\cdots(A-\lambda_k I)$$

we may show that $a_i=0\ \forall\ i=1,2,\ldots,k$

Thus the system of vectors $X_1, X_2, \ldots, X_k$ is linearly independent.

2. Unitary Matrix — A matrix A is said to be unitary if

$$A^{\theta} A = I$$

The transformation $X = AY$ is said to be unitary, if A is a unitary matrix.

Obviously a unitary matrix A is non-singular, its inverse A^{-1} being A^{θ}. Also, therefore

$$A A^{\theta} = I$$

3. Orthogonal Matrix — A matrix A is said to be orthogonal if it is real and

$$A' A = I$$

Clearly every orthogonal matrix is unitary.

The transformation $X = AY$ is said to be orthogonal if A is an orthogonal matrix.

The inverse of an orthogonal matrix A is A'. Therefore, orthogonal matrices are always non-singular and so. $AA' = I$

This proves that the transpose of an orthogonal matrix is also orthogonal.

Theorem-11 If A is a unitary matrix then A^{-1} is also a unitary

Proof— Given A is unitary $\therefore A A^{\theta} = I$

$$\Rightarrow (A A^{\theta})^{-1} = I^{-1}$$

$$\Rightarrow (A^{\theta})^{-1} A^{-1} = I$$

$$\Rightarrow (A^{-1})^{\theta} A^{-1} = I$$

$\Rightarrow A^{-1}$ is unitary.

Theorem-12 The products of two orthogonal matrices of the same order are orthogonal. Also the inverse of an orthogonal matrix is orthogonal.

Proof- Let A and B be two orthogonal matrices of the same order so that

$$AA' = I \quad \text{and} \quad BB' = I$$

Now we have

$$(AB)(AB)' = (AB)(B'A')$$
$$= A(BB')A'$$
$$= AIA' = AA' = I$$

Similarly, $(BA)(BA)' = I$

Therefore, AB and BA are also orthogonal matrices.

Also $AA' = I$

$$\Rightarrow (AA')^{-1} = I^{-1}$$
$$\Rightarrow (A')^{-1}A^{-1} = I$$
$$\Rightarrow (A^{-1})'A^{-1} = I$$
$$\Rightarrow A^{-1} \text{ is orthogonal.}$$

Cor- The products of two unitary matrices of the same order are unitary.

Theorem-13 The modulus of each characteristic root of a unitary matrix is unity.

Proof- Let λ be the characteristic root of a unitary matrix A then $AX = \lambda X$, $X \neq 0$

Taking conjugate transpose of each side, we have

$$X^{\Theta} A^{\Theta} = \bar{\lambda} X^{\Theta}$$

or $\quad X^{\theta} A^{\theta} A X = \bar{\lambda}\lambda X^{\theta} X \qquad (1)$

As A is unitary $\therefore A^{\theta} A = I$.

Now (1) is

$$X^{\theta} X = \bar{\lambda}\lambda X^{\theta} X$$

$$\Rightarrow \bar{\lambda}\lambda = 1$$

So that the modulus of λ is unity.

$\lambda = x + iy$
$\bar{\lambda} = x - iy$
$\therefore \lambda\bar{\lambda} = x^2 + y^2$
Also
$|\lambda| = |\bar{\lambda}| = \sqrt{x^2 + y^2}$
$= \sqrt{\lambda\bar{\lambda}} = \sqrt{1}$
$= 1$

Theorem-14 The modulus of each characteristic root of an orthogonal matrix is unity.

Proof- Let λ be the characteristic root of an orthogonal matrix A so that there exist a characteristic vector $X \neq 0$ such that

$$AX = \lambda X$$

Taking transpose of each sides, we get

$$X'A' = \lambda X'$$

or $\quad X'A'AX = \lambda X' \lambda X$

or $\quad X'IX = \lambda^2 X'X \qquad \because$ A is orthogonal

or $\quad X'X = \lambda^2 X'X$

or $\quad (1-\lambda^2) X'X = 0$

or $\quad (1-\lambda^2) = 0 \qquad \because X \neq 0 \Rightarrow X'X \neq 0$

or $\quad \lambda^2 = 1 \Rightarrow \lambda = \pm 1$

$\Rightarrow |\lambda| = 1$

Theorem-15 Every orthogonal matrix A can be expressed as

$$(I+S)(I-S)^{-1}$$

by a suitable choice of a real skew symmetric matrix S, provided that -1 is not a characteristic root of A.

Proof-- $A = (I+S)(I-S)^{-1}$

or $A(I-S) = (I+S)$

or $AI - AS = I+S$

or $AI - I = AS+S$

or $A - I = (A+I)S \qquad (1)$

Since -1 is not a characteristic root of A, so -1 will not satisfy the eqn $|A-\lambda I| = 0$

i.e. $|A+I| \neq 0$

i.e $(A+I)$ is non-singular

Now (1) gives

$$S = (A+I)^{-1}(A-I)$$

To show that S is real skew symmetric, we have to show $S' = -S$ & $\overline{S} = S$

Consider $S' = (A-I)'[(A+I)^{-1}]'$

$= (A-I)'[(A+I)']^{-1}$

$= (A'-I)(A'+I)^{-1}$

$= (A'+I)^{-1}(A'-I)$ as $(A'-I)$ & $(A'+I)$ is Commutative

$= (A'+A'A)^{-1}(A'-A'A)$ $\because$ A is orthogonal

$= [A'(I+A)]^{-1}[A'(I-A)]$

$= (I+A)^{-1}(A')^{-1}A'(I-A)$

$= (I+A)^{-1}(I-A) = -(A+I)^{-1}(A-I)$

$= -S$ $\therefore$ S is skew symmetric.

Now $\overline{S} = \overline{(A+I)^{-1}}\,\overline{(A-I)} = (\overline{A}+I)^{-1}(\overline{A}-I)$

$= (A+I)^{-1}(A-I) = S$ $\because$ A is orthogonal so its elements will be real

Theorem-16 If S is a real skew symmetric matrix, then $I-S$ is non-singular and $A = (I+S)(I-S)^{-1}$ is orthogonal

Proof- Since S is a real skew-symmetric matrix, therefore the characteristic roots of S are either zero or pure imaginary.

i.e the roots of the equation $|S-\lambda I|=0$ are either zero or pure imaginary number. Therefore 1 is not a root of the equation $|S-\lambda I|=0$

$$\text{So } |S-I| \neq 0$$

$\Rightarrow (S-I)$ is non-singular

$\Rightarrow (I-S)$ is non-singular

Now

$$A=(I+S)(I-S)^{-1} \Rightarrow A'=[(I+S)(I-S)^{-1}]'$$

$$\Rightarrow A' = [(I-S)^{-1}]'(I+S)'$$

$$\Rightarrow A' = [(I-S)']^{-1}(I+S)'$$

$$= (I'-S')^{-1}(I'+S')$$

$$= (I+S)^{-1}(I-S) \quad \because S \text{ is skew symm. } \therefore -S'=S$$

Therefore,

$$AA' = (I+S)(I-S)^{-1}(I+S)^{-1}(I-S)$$

$$= (I+S)(I+S)^{-1}(I-S)^{-1}(I-S) \quad \because (I-S)^{-1} \,\&\, (I+S)^{-1} \text{ are commutative.}$$

$$= I.I = I$$

Thus A is orthogonal.

4. <u>Inner Product of Two Vectors</u>– Let X and Y be two complex n-vectors written as column vectors. Suppose

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ and } Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

then the inner product of X and Y is defined as

$$X^{(i)} Y = \bar{x}_1 y_1 + \bar{x}_2 y_2 + \cdots + \bar{x}_n y_n$$

where $\bar{x}_i$ is the conjugate complex of the complex number x_i

It may be noted that the inner product of X and Y is not the same as that of Y and X i.e

$$X^{\circledast} Y \neq Y^{\circledast} X$$

If X and Y are real n-vectors written as column vectors then their inner product is defined as

$$X^{\circledast} Y = X'Y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$
$$= Y'X = Y^{\circledast} X$$

Thus in case of real vectors, the inner product of X and Y is same as that of Y and X.

5. <u>Length of a Vector</u>— Let X be a complex n-vector, then the positive square root of the inner product of X and X i.e $X^{\circledast} X$ is called the length of vector X. The length of a vector X is sometimes also called the norm of the vector X and is denoted by $\|X\|$.

A vector whose length is 1, is called a <u>normal vector</u> or <u>unit vector</u>.

6. <u>Orthogonal Vectors</u> — A vector X is said to be orthogonal to a vector Y, if the inner product of X and Y is zero i.e $X^{\circledast} Y = 0 \Leftrightarrow Y^{\circledast} X = 0$

i.e X is orthogonal to Y iff Y is orthogonal to X. On account of this property of symmetry it will be better to say that two vectors X and Y are orthogonal instead of saying one is orthogonal to other.

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Theorem-17 Any two characteristic vectors corresponding to two distinct characteristic roots of a Hermitian matrix are orthogonal.

Proof.- Let X_1 and X_2 be two characteristic vectors corresponding to two distinct characteristic roots λ_1 and λ_2 of a Hermitian matrix A.

$\therefore \quad AX_1 = \lambda_1 X_1 \qquad (1)$

and $\quad AX_2 = \lambda_2 X_2 \qquad (2)$

The roots λ_1 and λ_2 are real (See 16.7)

Now, by using (1) and (2)

$$X_2^{\theta} A X_1 = \lambda_1 X_2^{\theta} X_1$$
$$X_1^{\theta} A X_2 = \lambda_2 X_1^{\theta} X_2$$

But $(X_2^{\theta} A X_1)^{\theta} = X_1^{\theta} A X_2 \qquad$ for $A^{\theta} = A$

$\Rightarrow (\lambda_1 X_2^{\theta} X_1)^{\theta} = \lambda_2 X_1^{\theta} X_2$

$\Rightarrow \lambda_1 X_1^{\theta} X_2 = \lambda_2 X_1^{\theta} X_2$

$\Rightarrow (\lambda_1 - \lambda_2) X_1^{\theta} X_2 = 0$

$\Rightarrow X_1^{\theta} X_2 = 0 \quad$, as $\lambda_1 \neq \lambda_2$

Thus the vectors X_1 and X_2 are orthogonal.

Theorem-18 Any two characteristic vectors corresponding to two distinct characteristic roots of a real symmetric matrix are orthogonal.

Proof.- If the elements of a Hermitian matrix A are all real then A is a real symmetric. Thus a real symmetric matrix is Hermitian and therefore the result follows by previous theorem.

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<u>Theorem-19</u> Any two characteristic vectors, corresponding to two distinct characteristic roots of a unitary matrix are orthogonal.

<u>Proof—</u> Let $AX_1 = \lambda_1 X_1$ (1)

and $AX_2 = \lambda_2 X_2$ (2)

where $\lambda_1 \neq \lambda_2$

Taking conjugate transpose of (2), we get

$$X_2^{\Theta} A^{\Theta} = \bar{\lambda}_2 X_2^{\Theta}$$

$$\Rightarrow \quad X_2^{\Theta} A^{\Theta} A X_1 = \bar{\lambda}_2 \lambda_1 X_2^{\Theta} X_1$$

$$\Rightarrow \quad X_2^{\Theta} X_1 = \bar{\lambda}_2 \lambda_1 X_2^{\Theta} X_1 \quad , \text{ for } A^{\Theta} A = I$$

$$\Rightarrow \quad (1 - \bar{\lambda}_2 \lambda_1) X_2^{\Theta} X_1 = 0$$

As A is unitary matrix, the modulus of each of its characteristic roots is unity, so that

$$\bar{\lambda}_2 \lambda_2 = 1$$

Therefore,

$$(1 - \bar{\lambda}_2 \lambda_1) = \bar{\lambda}_2 \lambda_2 - \bar{\lambda}_2 \lambda_1$$

$$= \bar{\lambda}_2 (\lambda_2 - \lambda_1) \neq 0$$

$$\therefore \quad X_2^{\Theta} X_1 = 0$$

i.e. the vectors X_2 and X_1 are orthogonal.

<u>Theorem-20</u> Every orthogonal set of non-zero vectors is linearly independent.

<u>Proof—</u> Let $X = \{X_1, X_2, \cdots, X_k\}$ be an orthogonal set of non-zero complex n-vectors. Then to prove that X is linearly independent.

Let $c_1, c_2, \cdots, c_k$ be scalars such that

$$C_1X_1 + C_2X_2 + \cdots + C_kX_k = 0 \qquad (1)$$

Let $1 \le m \le k$, then forming inner product of both sides of (1) with the vector X_m, we get:

$$(X_m,\ C_1X_1 + C_2X_2 + \cdots + C_kX_k) = (X_m, 0)$$

or $C_1(X_m, X_1) + C_2(X_m, X_2) + \cdots + C_k(X_m, X_k) = 0$ $\quad \because (X_m, 0) = 0$

or $C_m(X_m, X_m) = 0$, $\because$ any two distinct vectors of X are orthogonal

or $C_m = 0$, since $X_m \neq 0 \Rightarrow (X_m, X_m) \neq 0$

Thus $C_m = 0$, for $m = 1, 2, \cdots k$. In this way the relation (1) implies that $C_1 = 0, C_2 = 0, \cdots C_k = 0$. Therefore the set of vectors $X_1, X_2, \cdots X_k$ is linearly independent.

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Exercise-2 Show that the matrix

$$A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

satisfies Cayley-Hamilton theorem.

Exercise-3 State Cayley-Hamilton theorem. Use it to express $2A^5 - 3A^4 + A^2 - 4I$ as a linear polynomial in A when $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

Solⁿ– Cayley Hamilton theorem - Every square matrix satisfies its characteristic equation.

Now let us find the characteristic equation of the matrix A. We have

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(2-\lambda) + 1 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 7 = 0 \qquad (1)$$

By Cayley - Hamilton theorem, the matrix A must satisfy (1). Therefore,

$$A^2 - 5A + 7I = 0 \qquad (2)$$

$$\Rightarrow A^2 = 5A - 7I \qquad (3)$$

and $A^3 = 5A^2 - 7A$

or $A^3 = 5(5A - 7I) - 7A$

or $A^3 = 18A - 35I \qquad (4)$

Now $A^4 = 18A^2 - 35A$

$= 18(5A - 7I) - 35A$

$= 55A - 126I \qquad (5)$

Therefore

$$2A^5 - 3A^4 + A^2 - 4I = 2(55A^2 - 126A) - 3(55A - 126I) + 5A - 7I - 4I$$

$$= 110(5A - 7I) - 252A - 165A + 378I + 5A - 11I$$

$$= 550A - 770I - 252A - 165A + 378I + 5A - 11I$$

$$= 138A - 403$$

which is a linear polynomial in A

<u>Note</u> - Converse of Th 20 is not true, as $x_1 = (1, 2, 3)$ and $(2, 2, 0)$ are linearly independent but not orthogonal.

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