3.6.2 Solved Problems on Pole-zero Plot :

Ex. 3.6.1 : **How the characteristic behaviour of causal discrete time signals depends on pole-zero location with respect to unit circle ? Explain with examples.**

Soln. : A causal LSI system is a system present at positive values of Z. All the characteristics of such system depends on pole-zero location with respect to unit circle. We will consider few examples.

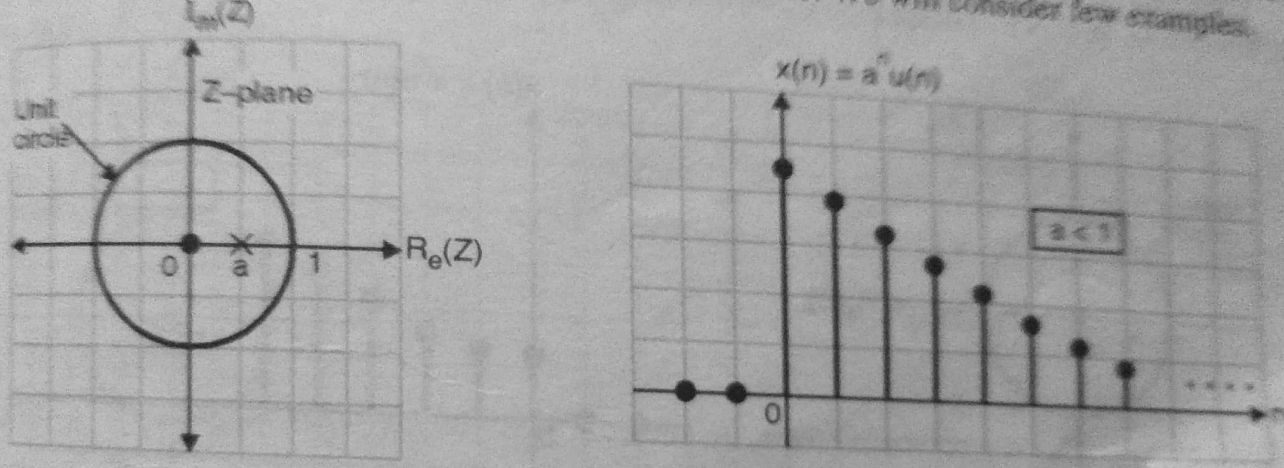

Fig. P. 3.6.1(a) : Pole-zero plot of $a^n u(n)$

(i) Let $x(n) = a^n u(n)$

If a < 1, we will get decaying exponential. Its Z-transform is,

$$X(Z) = \frac{Z}{Z-a} = \frac{(Z-0)}{Z-a} \quad \ldots(1)$$

Thus zero is at Z = 0 (origin) and poles is at Z = a. It is shown in Fig. P. 3.6.1(a).

Here zero is marked by '0' and pole is marked by 'x'. As shown in Fig. P. 3.6.1(a), the pole inside unit circle.

(ii) Let $x(n) = u(n)$

Then $X(Z) = \frac{Z}{Z-1} = \frac{(Z-0)}{(Z-1)}$

Thus zero is at $Z = 0$ (origin) and pole is at $Z = 1$ (on the unit circle.) It is shown in, Fig. P. 3.6.1(b).

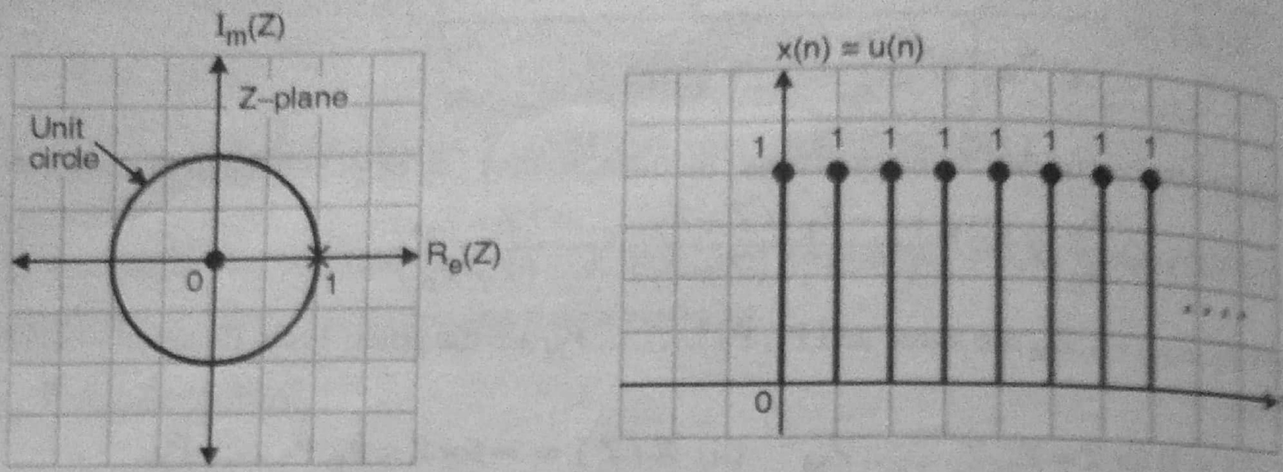

Fig. P. 3.6.1(b) : Pole-zero plot of u(n)

(iii) Let $x(n) = a^n u(n)$ and $a > 1$ then we will get rising exponential.

Now $X(Z) = \frac{Z}{Z-a} = \frac{Z-0}{Z-a}$

Thus zero is at '0' (origin) and pole is at 'a'. But in this case $a > 1$. So pole is outside the unit circle. It is shown in Fig. P. 3.6.1(c).

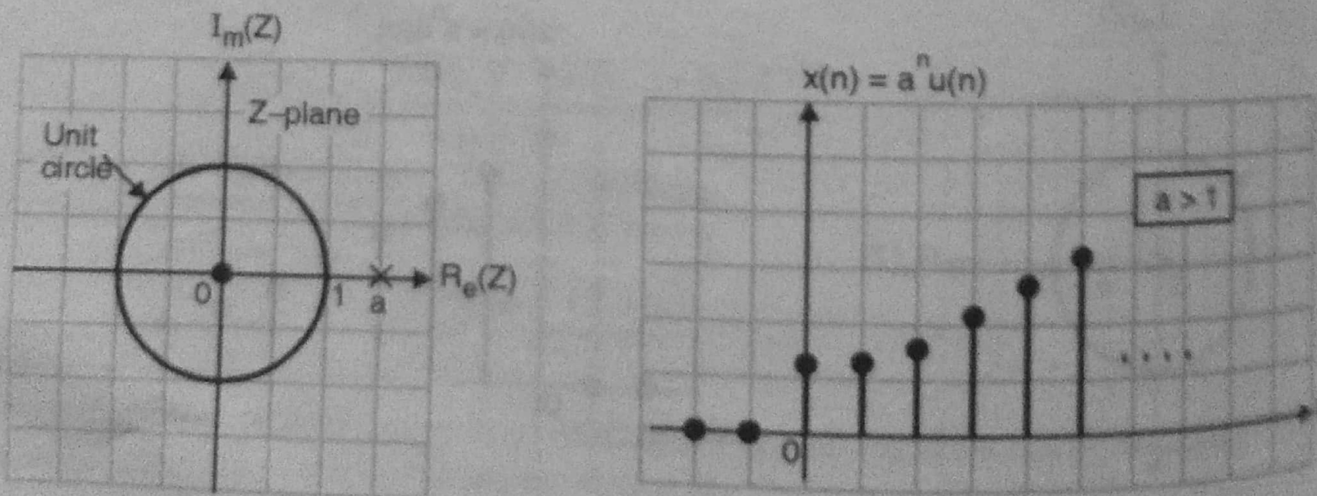

Fig. P. 3.6.1(c)

(iv) Let $x(n) = -a^n u(n)$ and $a < 1$

then $X(Z) = -\frac{Z}{Z-a} = \frac{Z}{-Z+a}$

Thus zero is at Z = '0' (origin) and pole is at $-Z + a = 0$ that means $Z = -a$. This pole-zero plot is shown in Fig. P. 3.6.1(d).

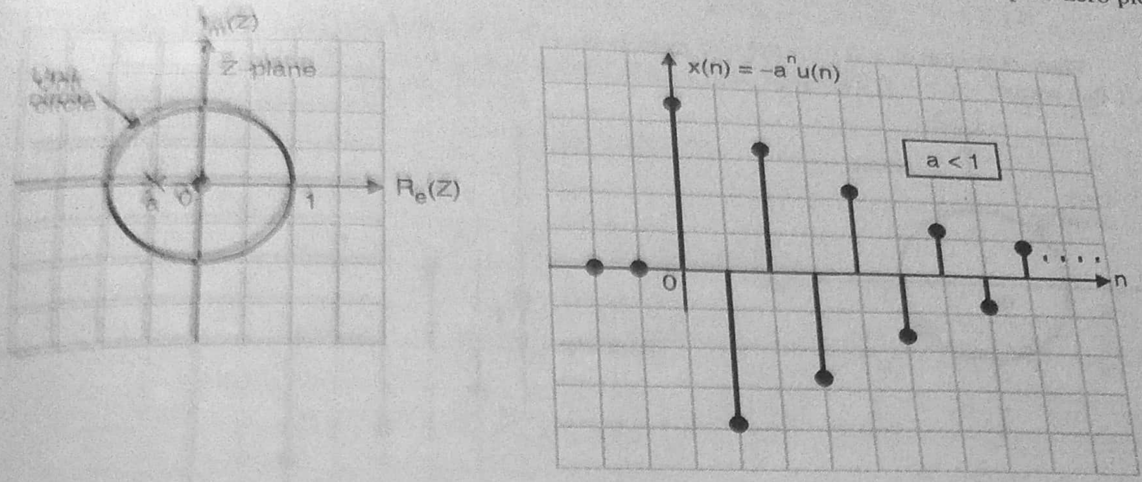

Fig. P. 3.6.1(d) : $x(n) = -a^n u(n)$, if $a < 1$

(e) Let $x(n) = (-1)^n u(n)$, we have standard Z-transform pair,

$$a^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-a}$$

Let $a = -1$

$$\therefore (-1)^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z+1} \qquad \ldots(5)$$

Thus zero is at Z = '0' (origin) and pole is at $Z + 1 = 0$ that means $Z = -1$. This pole zero plot is shown in Fig. P. 3.6.1(e).

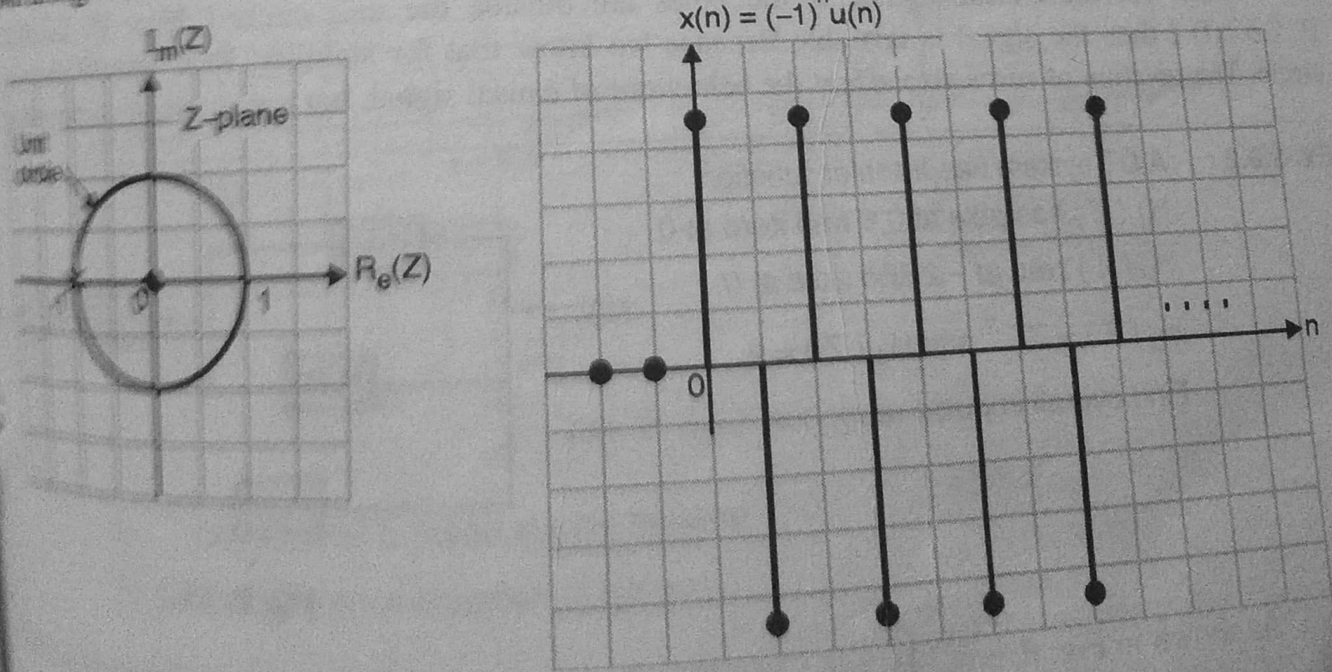

Fig. P. 3.6.1(e) : $x(n) = (-1)^n u(n)$

(f) Let $x(n) = -a^n u(n)$ and $a > 1$.

$$\therefore\ X(Z) = -\frac{Z}{Z-a} = \frac{Z}{-Z+a} \qquad ...(6)$$

Thus zero is at $Z = 0$ (origin) and pole is at $-Z + a = 0$ that means at $Z = -a$. But in this case $a > 1$ that means $-a > -1$. This pole zero plot is shown in Fig. P. 3.6.1(f).

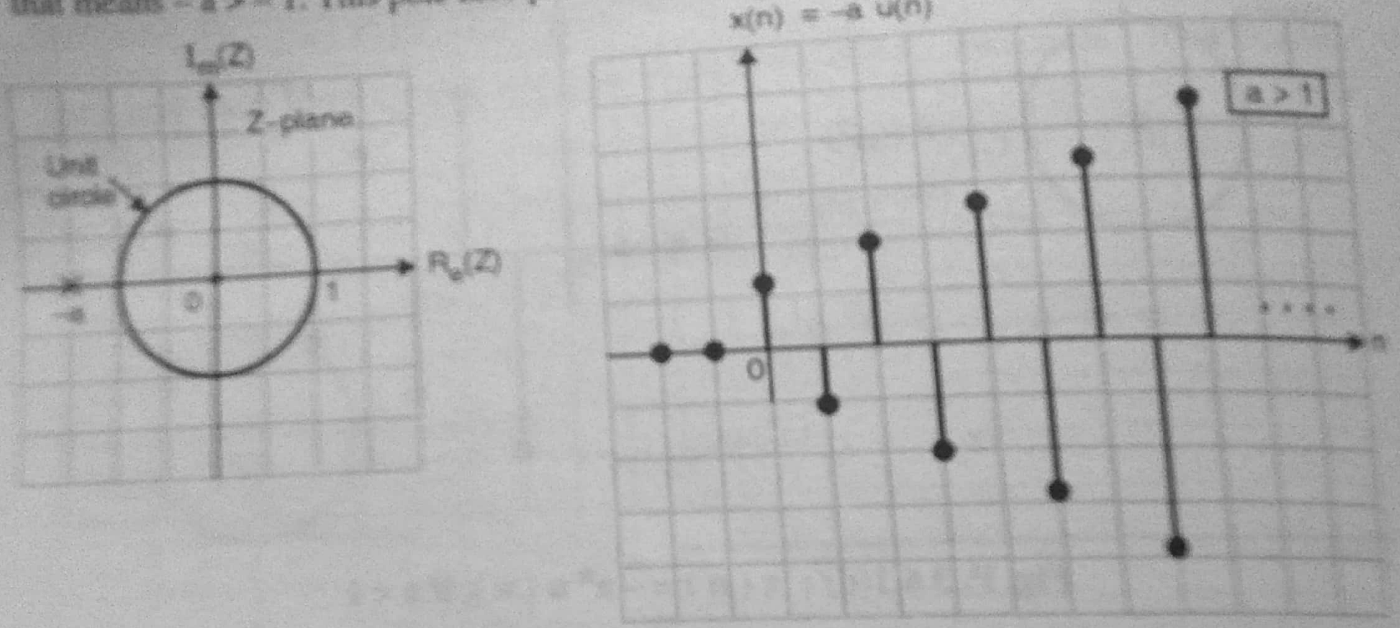

Fig. P. 3.6.1(f) : $x(n) = -a^n u(n)$ if $a > 1$

From this discussion we can conclude the following points :

(1) If the pole is inside unit circle then the signal decays [Fig. P. 3.6.1(a)].

(2) If the pole is on the unit circle then the signal has fixed amplitude [Fig. P. 3.6.1(b)].

(3) If the pole is outside the unit circle then the amplitude of signal is increasing [Fig. P. 3.6.1(c)

(4) If there is negative pole then the signal alters in sign as shown in Figs. P. 3.6.1(d), P. 3.6.1(e) P. 3.6.1(f).

Also for the causal signals if the poles are outside the unit circle [Figs. P. 3.6.1(c) P. 3.6.1(f)] then the signal is unstable.Because we know that for stability; ROC should include circle. The position of zeros also affect the behaviour of causal signal, but not as strong as the poles

Ex. 3.6.2 : A DT system has transfer function

$H_1(Z)$ has pole at 0.6 and zero at 0.

$H_2(Z)$ zero at -2 and pole at 0.

$H_3(Z) = Z^{-1}$ and $H_4(Z) = 2$.

Find overall impulse response.

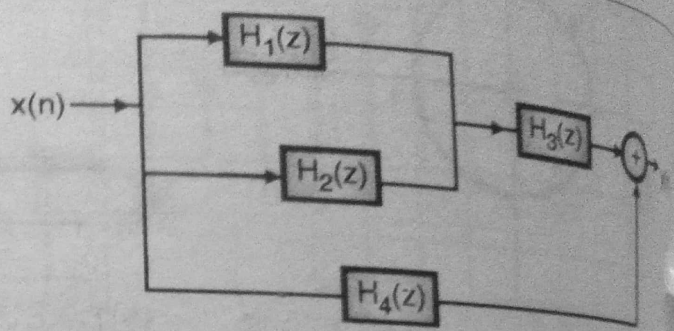

Fig. P. 3.6.2

Soln. : As shown in Fig. P. 3.6.2 $H_1(Z)$ and $H_2(Z)$ are in parallel and this parallel combination is series with $H_3(Z)$. While $H_4(Z)$ is parallel to this combination. Thus overall transfer function is,

$$H(Z) = \{[H_1(Z) + H_2(Z)] \cdot H_3(Z)\} + H_4(Z)$$

$H_1(Z)$ has pole at 0.6 and zero at 0.

$$H_1(Z) = \frac{Z}{Z-0.6}$$

$\therefore$

$H_2(Z)$ has pole at origin and zero at -2. ...(2)

$$H_2(Z) = \frac{Z+2}{Z}$$

$\therefore$ ...(3)

$$H_3(Z) = Z^{-1} \text{ and } H_4(Z) = 2.$$

Putting these values in Equation (1).

$$H(Z) = \left[\left(\frac{Z}{Z-0.6} + \frac{Z+2}{Z}\right) \cdot Z^{-1}\right] + 2$$

$$\therefore H(Z) = \frac{1}{Z-0.6} + \frac{1+2Z^{-1}}{Z} + 2 = Z^{-1}\left(\frac{Z}{Z-0.6}\right) + \frac{1}{Z} + \frac{2}{Z^2} + 2$$

$$\therefore H(Z) = Z^{-1}\left(\frac{Z}{Z-0.6}\right) + Z^{-1} + Z^{-2} \cdot 2 + 2$$

We have standard Z-transform pairs,

$$(\alpha)^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \text{ and } \delta(n) \overset{Z}{\longleftrightarrow} 1$$

$$\therefore h(n) = (0.6)^{n-1} \cdot u(n-1) + \delta(n-1) + 2\delta(n-2) + 2\delta(n)$$

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3.6.3: A causal DT system has transfer function $H(Z)$ such that $H(Z) = H_1(Z) \cdot H_2(Z)$ The pole-zero diagram is as follows :

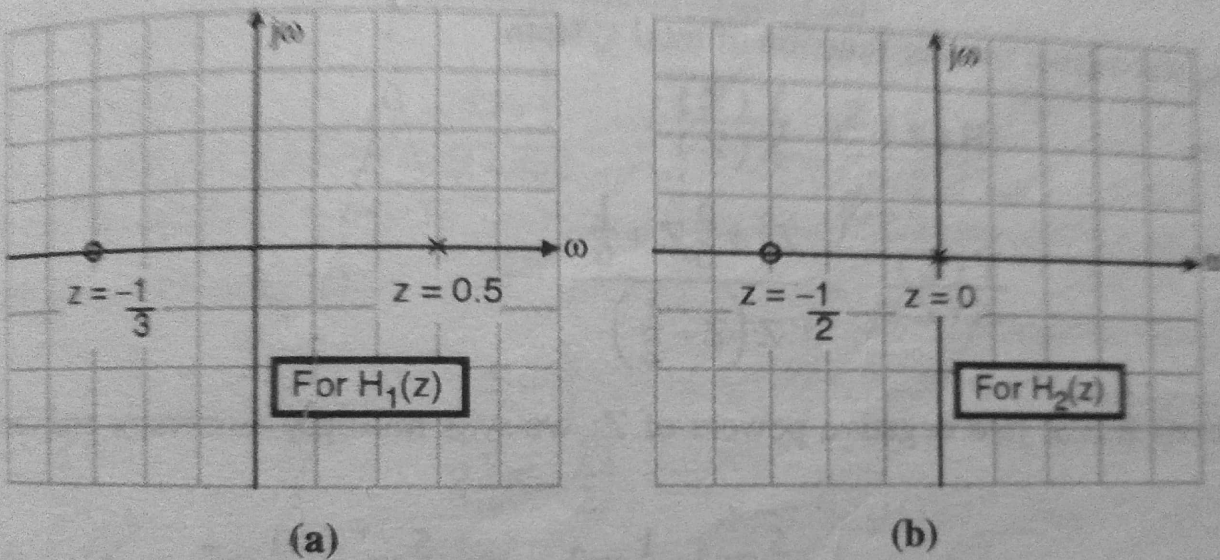

Fig. P. 3.6.3

(i) Find transfer function of total system.

(ii) Find difference equation of system.

(iii) Find response of system to input $x(n) = \left(-\frac{1}{2}\right)^n u(n)$

Soln. :

(i) $H_1(Z)$ has pole at $Z = 0.5$ and zero at $Z = -\frac{1}{3}$.

We have, $$H_1(Z) = \frac{Z - Z_1}{Z - P_1}$$

$$\therefore \quad H_1(Z) = \frac{Z + 1/3}{Z - 0.5} \quad \text{...(1)}$$

$H_2(Z)$ has zero at $Z = -1/2$ and pole at $Z = 0$.

$$\therefore \quad H_2(Z) = \frac{Z - Z_1}{Z - P_1}$$

$$\therefore \quad H_2(Z) = \frac{Z + 1/2}{Z} \quad \text{...(2)}$$

Given $$H(Z) = H_1(Z) \cdot H_2(Z)$$

$$\therefore \quad H(Z) = \frac{Z + \frac{1}{3}}{Z - 0.5} \cdot \frac{Z + \frac{1}{2}}{Z} \quad \text{...(3)}$$

$$\therefore \quad H(Z) = \frac{Z^2 + \frac{1}{2}Z + \frac{1}{3}Z + \frac{1}{6}}{Z(Z - 0.5)}$$

$$\therefore \quad H(Z) = \frac{Z^2 + \frac{5}{6}Z + \frac{1}{6}}{Z\left(Z - \frac{1}{2}\right)}$$

This equation gives transfer function of total system.

(ii) We have $$H(Z) = \frac{Y(Z)}{X(Z)}$$

$$\therefore \quad \frac{Y(Z)}{X(Z)} = \frac{Z^2 + \frac{5}{6}Z + \frac{1}{6}}{Z\left(Z - \frac{1}{2}\right)} \quad \text{...(4)}$$

To convert R.H.S. into negative powers of Z; we will multiply numerator and denominator by Z^{-2}.

$$\therefore \quad \frac{Y(Z)}{X(Z)} = \frac{1 + \frac{5}{6}Z^{-1} + \frac{1}{6}Z^{-2}}{Z^{-1}\left(Z - \frac{1}{2}\right)} = \frac{1 + \frac{5}{6}Z^{-1} + \frac{1}{6}Z^{-2}}{1 - \frac{1}{2}Z^{-1}}$$

$$\therefore \quad Y(Z)\left[1 - \frac{1}{2}Z^{-1}\right] = X(Z)\left[1 + \frac{5}{6}Z^{-1} + \frac{1}{6}Z^{-2}\right]$$

$$\therefore \quad Y(Z) - \frac{1}{2}Z^{-1}Y(Z) = X(Z) + \frac{5}{6}Z^{-1}X(Z) + \frac{1}{6}Z^{-2}X(Z)$$

Taking IZT of both sides we get,

$$y(n)-\frac{1}{2}y(n-1)=x(n)+\frac{5}{6}x(n-1)+\frac{1}{6}x(n-2)$$ ...(5)

This is the difference equation of a system.

(iii) Given input is,

$$x(n) = \left(-\frac{1}{2}\right)^n u(n)$$

We have standard Z-transform pair.

$$a^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-a}$$

Here $a = -\frac{1}{2}$

$$\therefore X(Z) = \frac{Z}{Z+\frac{1}{2}}$$ ...(6)

Now $$H(Z) = \frac{Y(Z)}{X(Z)}$$

$$\therefore Y(Z) = H(Z)\cdot X(Z)$$ ...(7)

Putting Equations (3) and (6) in Equation (7)

$$Y(Z) = \left[\frac{Z+\frac{1}{3}}{Z-0.5}\cdot\frac{Z+\frac{1}{2}}{Z}\right]\cdot\frac{Z}{Z+\frac{1}{2}}$$

$$\therefore Y(Z) = \frac{Z+\frac{1}{3}}{Z-0.5} \quad \therefore Y(Z) = \frac{Z+\frac{1}{3}}{Z-\frac{1}{2}}$$

We will obtain IZT using P.F.E.

$$\frac{Y(Z)}{Z} = \frac{Z+\frac{1}{3}}{Z\left(Z-\frac{1}{2}\right)}$$

In PFE form Equation (8) can be written as,

$$\frac{Y(Z)}{Z} = \frac{A_1}{Z}+\frac{A_2}{Z-\frac{1}{2}}$$

$$A_1 = (Z-0)\cdot\left.\frac{Z+\frac{1}{3}}{Z\left(Z-\frac{1}{2}\right)}\right|_{Z=0} = \frac{1/3}{-1/2} = -\frac{2}{3}$$

$$\text{and } A_2 = \left(Z-\frac{1}{2}\right)\cdot\left.\frac{Z+\frac{1}{3}}{Z\left(Z-\frac{1}{2}\right)}\right|_{Z=\frac{1}{2}}$$

$$\therefore \quad A_2 = \frac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{2}} = \frac{5}{6}\times 2 = \frac{5}{3}$$

Thus Equation (9) becomes,

$$\frac{Y(Z)}{Z} = \frac{-2/3}{Z} + \frac{5/3}{Z-\frac{1}{2}}$$

$$\therefore \; Y(Z) = -\frac{2}{3} + \frac{5}{3}\cdot\frac{Z}{Z-\frac{1}{2}}$$

IZT of Equation (10) can be written as,

$$\boxed{y(n) = -\frac{2}{3}\delta(n) + \frac{5}{3}\left(\frac{1}{2}\right)^n u(n)}$$

This is the response of system.