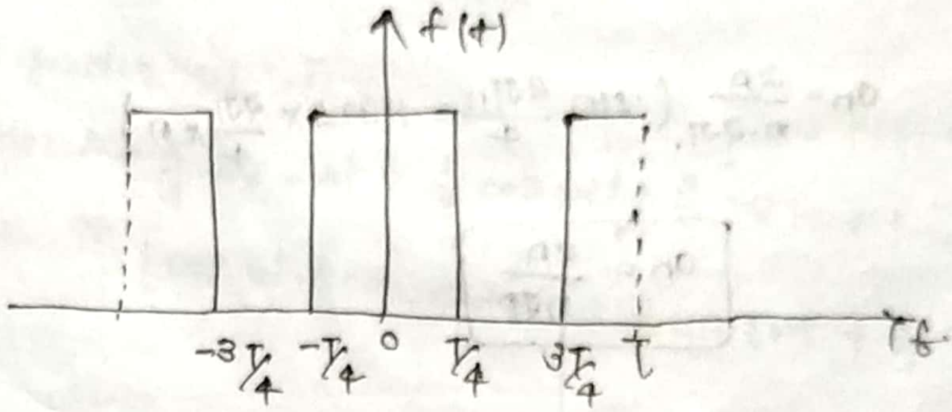
# Fourier series.

1. Q4 Find the fourier series expansion of the periodic rectangular wave form show in fig.

$$f(t) = A, \quad 0 < t < T/4$$
$$= 0, \quad T/4 < t < 3T/4$$
$$= A, \quad 3T/4 < t < T$$




$$a_n = \frac{2}{T}\left[\int_0^{T/4} A \cdot \cos n\omega t \, dt + \int_{T/4}^{3T/4} 0 \cdot \cos n\omega t \, dt + \int_{3T/4}^{T} A \cdot \cos n\omega t \, dt\right].$$

$$= \frac{2A}{T}\left[\int_0^{T/4} \cos n\omega t \, dt + \int_{3T/4}^{T} \cos n\omega t \, dt\right].$$

$$= \frac{2A}{T}\left[\left\{\frac{\sin n\omega t}{n\omega}\right\}_0^{T/4} + \left\{\frac{\sin n\omega t}{n\omega}\right\}_{3T/4}^{T}\right]$$

$$= \frac{2A}{n\omega T}\left[\sin n\frac{\omega T}{4} + 0 + \sin n\omega T - \sin\frac{3n\omega T}{4}\right].$$

$$= \frac{2A}{n\omega T}\left[\sin\frac{n\omega T}{4} + \sin 2\pi n - \sin\frac{3n\omega T}{4}\right]$$

$\sin 2\pi n = 0.$

$$\boxed{a_n = \frac{2A}{n\omega T}\left[\sin\frac{n\omega T}{4} - \sin\frac{3n\omega T}{4}\right].}$$

$\omega T = 2\pi$

$$a_n = \frac{2A}{n\cdot 2\pi}\left(\sin \frac{2\pi n}{4} - \sin 3\times\frac{2\pi}{4}\times n\right)$$

$$\boxed{a_n = \frac{2A}{n\pi}}$$

for all integer values of $n$,

$a_n = \frac{2A}{n\pi}$ , $n = 1, 5, 9.$

$a_n = \frac{-2A}{n\pi}$ , $n = 3, 7, 11$

$a_n = 0$ , $n$ even.

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t\, dt.$$

$$= \frac{2}{T}\left[\int_0^{T/4} A\cdot\sin n\omega t\, dt + \int_{T/4}^{3T/4} 0\cdot\sin n\omega t\, dt + \int_{3T/4}^{T} A\sin n\omega t\, dt\right].$$

$$b_n = \frac{2A}{n\omega T}\left[-\frac{\cos n\omega T}{4} + 1 \cdot \cos n\omega T + \frac{\cos 3n\omega T}{4}\right].$$

$$\omega T = 2\pi$$

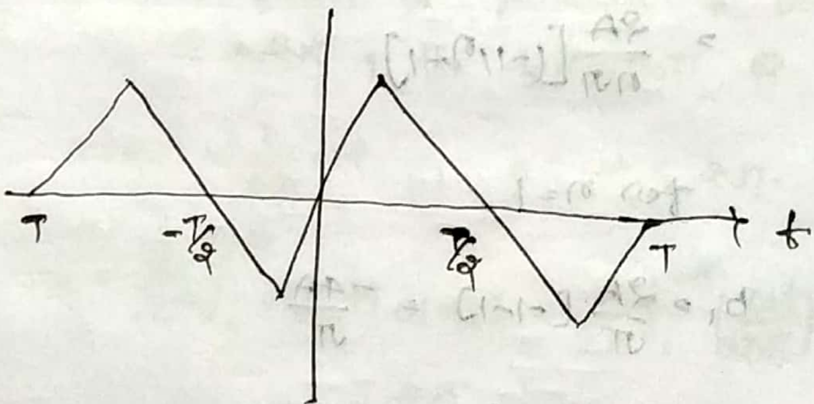
$$= \frac{2A}{2\pi n}\left[1 + (-1)\right].$$

$$b_n = 0.$$

The fourier series is thus obtained as $\to f(t) = \frac{A}{\pi}\left[2\cos\omega t + \frac{2}{3}\cos 3\omega t + \frac{2}{5}\cos 5\omega t + \frac{2}{7}\cos 7\omega t + \dots\right].$

$$f(t) = \frac{2A}{\pi}\left[\cos\omega t - \frac{1}{3}\cos 3\omega t + \frac{1}{5}\cos 5\omega t + \frac{1}{7}\cos 7\omega t\right]$$

Qu 4




the function being odd, $a_0 = 0$ & $a_n = 0$ the sine coefficient is given by.

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\sin(n\omega_0 t)\,dt$$

$$= \frac{4}{T}\int_0^{T/2} A\cdot\sin(n\omega_0 t)\,dt$$

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$$= \frac{4A}{T}\int_0^{T/2} \sin(n\omega_0 t)\,dt$$

$$= \frac{4A}{n\omega_0 T}\left[\cos(n\omega_0 t)\right]_0^{T/2}$$

$$= \frac{4A}{n\omega_0 T}\left[\cos\frac{n\omega_0 T}{2} - 1\right]$$

$$\omega T = 2\pi$$

$$b_n = \frac{4A}{n.2\pi}\left[\cos\frac{n.2\pi}{2} - 1\right].$$

$$= \frac{2A}{n\pi}\left[\cos n\pi - 1\right].$$

$$= \frac{2A}{n\pi}\left[(-1)^n - 1\right],$$

for $n = 1$

$$b_1 = \frac{2A}{\pi}[-1-1] = -\frac{4A}{\pi}$$

for $n = 2$

$$b_2 = \frac{2A}{\pi}[1-1] = 0$$

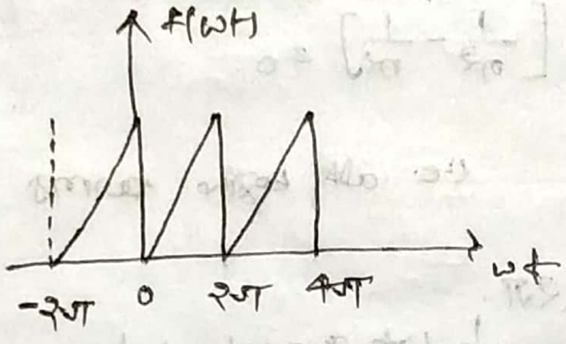
for $n = 3$

$$b_3 = \frac{2A}{3\pi}(-1-1) = -\frac{4A}{3\pi}$$

fourier series is given by

$$f_t = \frac{4A}{\pi}\left[-\sin\omega_0 t - \frac{1}{3}\sin 3\omega_0 t - \frac{1}{5}\sin 5\omega_0 t + \dots\right].$$

(Q4) Determine the fourier series for the sawtooth wave shown in fig.




the wave form is periodic & continuous for $0 < \omega t < 2\pi$, the function is defined as

$$f(\omega t) = \frac{1}{2\pi}\omega t \quad 0 < \omega t < 2\pi.$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\omega t)\, d(\omega t) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\omega t}{2\pi}\right) d(\omega t)$$

$$= \frac{1}{2\pi}\left[\frac{(\omega t)^2}{2\pi}\right]_0^{2\pi} = \frac{1}{2\pi}\times\left[\frac{(2\pi)^2}{4\pi} - 0\right].$$

$$= \frac{1}{2\pi}\times\frac{4\pi^2}{4\pi} = \frac{1}{2}$$

$$= 0.5$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\omega t\right)\cos n\omega t \, dt$$

$$= \frac{1}{2\pi^2}\left[\frac{\omega t}{n}\sin n\omega t + \frac{\cos n\omega t}{n^2}\right]_0^{2\pi}$$

Putting limits, we get,

$$= \frac{1}{2\pi^2}\left[\frac{2\pi}{n}\sin 2\pi n + \frac{\cos 2\pi n}{n^2} - 0 - \frac{\cos 0}{n^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{n^2} - \frac{1}{n^2}\right] = 0.$$

i.e. all cosine terms are absent

$$b_n = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2\pi}\omega t \sin \omega t \cdot d\omega t$$

$$= \frac{1}{2\pi^2}\left[\frac{-\omega t}{n}\cos n\omega t + \frac{1}{n^2}\sin \omega t\right]_0^{2\pi}$$

$$= \frac{1}{2\pi^2}\left[-\frac{(2\pi)}{n}\cos 2\pi n + \frac{1}{n^2}\sin 2\pi + 0 - 0\right]$$

$$= \frac{-2\pi}{2\pi^2 n} \quad \text{i.e.} \quad -\frac{1}{\pi n}.$$

Thus eq. (1) yield the Fourier series as.

$$f(t) = .5 - \sum_{n=1}^{\infty}\frac{1}{n\pi}\sin n\omega t$$

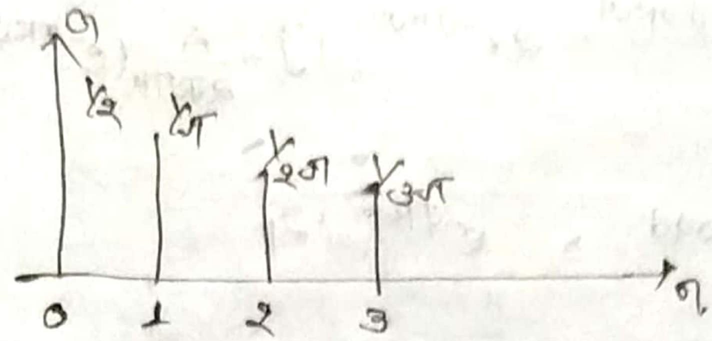
$$C_0 = a_0 = .5$$

$a_0 = .5$

$a_n = 0$

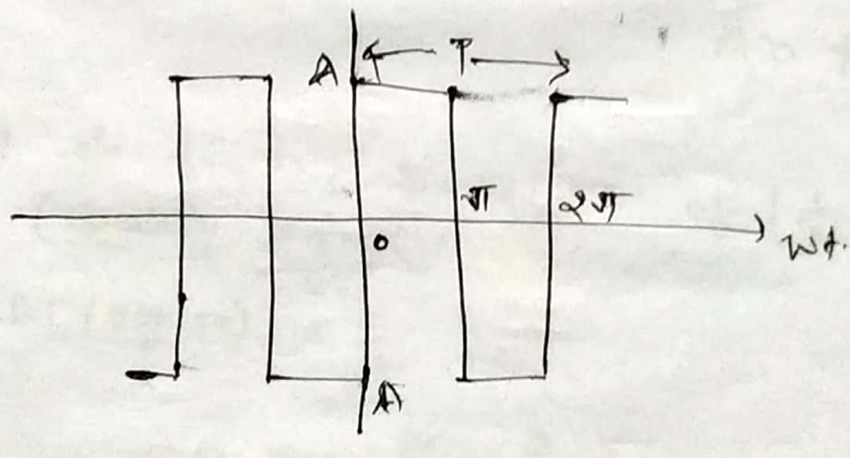
$b_n = \frac{1}{-\pi n}$

as $a_n = 0$ hence $C_n = |b_n| = \frac{1}{n\pi}$




magnitude spectrum of sawtooth wave

Ans,




$$C_n = \frac{1}{2\pi}\left[\int_0^{\pi} A e^{-jk(\omega t)}\, d(\omega t) + \int_{\pi}^{2\pi} (-A) e^{-jk(\omega t)}\, d(\omega t)\right]$$

$$= \frac{1}{2\pi}\left[\frac{A}{-jk}\left[e^{-jk(\omega t)}\right]_0^{\pi} + \frac{+A}{-jk}\left[e^{-jk(\omega t)}\right]_{\pi}^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{A}{-jk}\left(e^{-jk\pi} - 1\right) + \frac{A}{jk}\left(e^{-j2k\pi} - e^{-jk\pi}\right)\right]$$

$$c_n = \frac{A}{2\pi j k}\left[-1 - e^{-jk\pi} + e^{-j2k\pi} - e^{-jk\pi}\right].$$

$$= \frac{A}{2jk}\left[e^{-j2k\pi} - 2e^{-jk\pi} - 1\right] = \frac{A}{2j\pi k}\left(e^{-jk\pi} - 1\right).$$

for $n = \text{odd}$, $e^{-jk\pi} = -1$.

So,

$$c_n = \frac{A}{2j\pi k}\left(e^{-jk\pi} - 1\right)^2 = \frac{A}{2j\pi k}(-1-1)^2$$

$$\underset{n=\text{odd}}{c_n} = \frac{2A}{j\pi k}$$

$\frac{A}{2}$

# Fourier Transform

Q. Shown that if $x_3(t) = a x_1(t) + b x_2(t)$, then $x_3(\omega) = a x_1(\omega) + b x_2(\omega)$.

Sol

$$x_3(t) = a x_1(t) + b x_2(t) \quad \text{--- (i)}$$

taking fourier transform

$$X_3(\omega) = F[a x_1(t)] + F[b x_2(t)]$$

$$= \int_{-\infty}^{\infty} a x_1(t) e^{-j\omega t} dt + \int_{-\infty}^{\infty} b x_2(t) e^{-j\omega t} dt$$

$$= a \int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} dt + b \int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} dt$$

Thus $X_3(\omega) = a X_1(\omega) + b X_2(\omega)$

hence proved.

Q. Find the fourier transform of $\left(\frac{1}{jt}\right)$ (duality property)

Sol. $F.T. [\operatorname{sgn}(t)] = \frac{2}{j\omega}$

$$\operatorname{sgn} \xleftrightarrow{FT} \frac{2}{j\omega}$$

$$\frac{2}{jt} \longleftrightarrow 2\pi \operatorname{sgn}(-\omega)$$

Ans

(Q) FT $[e^{-at}u(t) * e^{-bt}u(t)]$

(S) $F[e^{-at}u(t)] = \frac{1}{a+j\omega}$ — (i)

$F[e^{-bt}u(t)] = \frac{1}{b+j\omega}$ — (ii)

By using convolution property.

$F.T.[(i) * (ii)] = \frac{1}{a+j\omega} * \frac{b}{b+j\omega}$ Ans

(Q) $F^{-1}\left[\frac{1}{a+j\omega} * \frac{2}{j\omega}\right]$

(S) $F^{-1}\left[\frac{1}{a+j\omega}\right] = e^{-at}u(t)$ — (i)

$F^{-1}\left[\frac{2}{j\omega}\right] = sgn(t)$

$F^{-1}\left[\frac{1}{a+j\omega}\right] * F^{-1}\left[\frac{2}{j\omega}\right] =$

$e^{-at}u(t) \cdot sgn(t)$

Ans