6.7. Reconstruction Filter (Low Pass Filter)

The low pass filter is used to recover original signal from its samples. This is also called interpolation filter.

A low-pass filter is that type of filter which passes only low-frequencies upto a specified cut-off frequency and rejects all other frequenices above cut-off frequency. Figure 6.11 shows the frequency response of a low-pass filter.

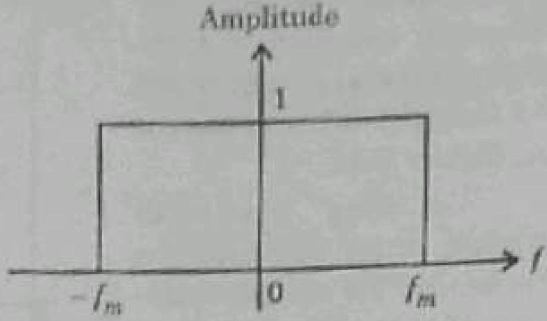

Fig. 6.11. Ideal low-pass filter.

From figure 6.11 it may be observed that in case of low-pass filter, there is sharp-change in response at cut-off frequency, that is amplitude response becomes suddenly zero at cut-off frequency which is not possible practicaliy. This means that an ideal low-pass filter is not physically realizable. In place of ideal-low pass filter, we use practical filter.

Figure 6.12 shows the frequency response of practical low-pass filter. From figure 6.12, it may be observed that in case of practical filter, the amplitude response decreases slowly to become zero. This means that there is a transition band in case of practical filter. Figure 6.13 shows the use of practical low-passfilter in reconstruction of original signal from its sample.

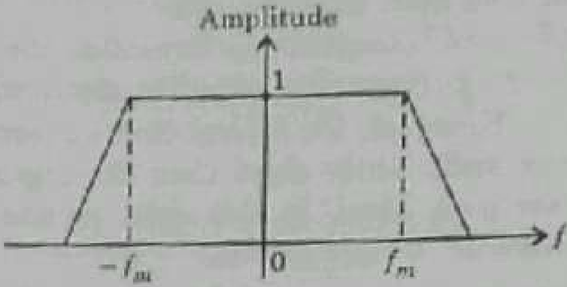

Fig. 6.12. Practical low-pass filter.

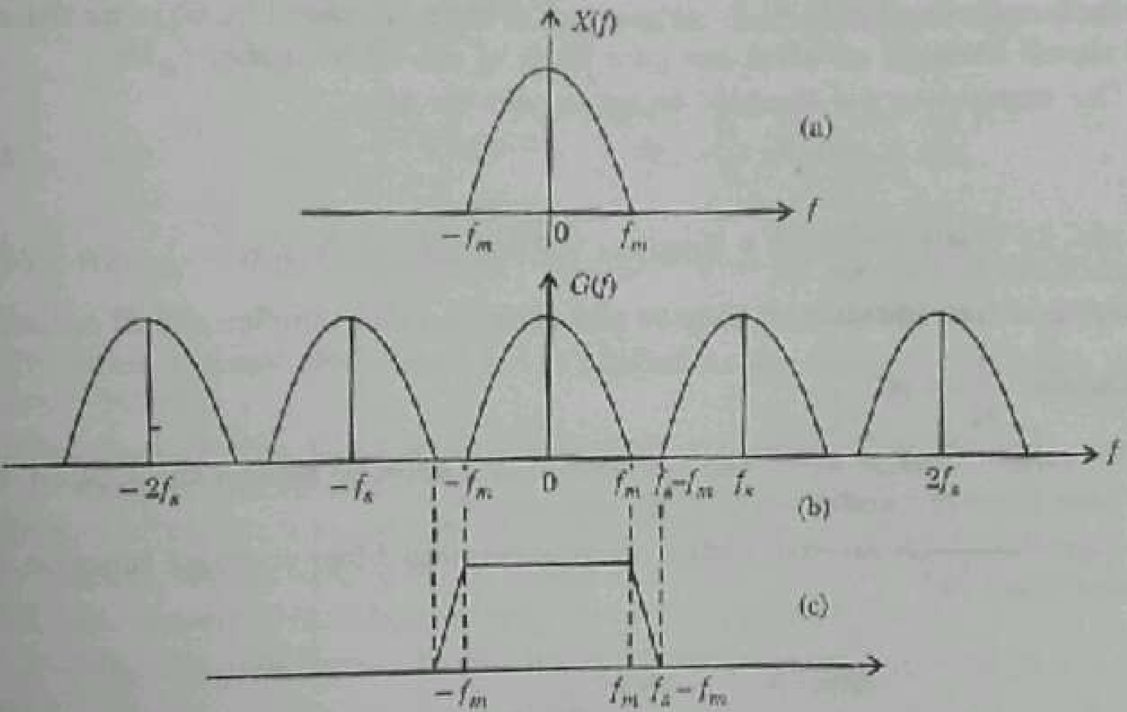

Fig. 6.13 (a) Spectrum of original signal
(b) Spectrum of sampled signal
(c) Amplitude response of practical low-pass filter.

6.8. Signal Reconstruction : The Interpolation Formula

(U.P. Tech., Semester, Examination, 2003-04)

The process of reconstructing a continuous-time signal $x(t)$ from its samples is known as **interpolation.**

Interpolation is the commonly used procedure for reconstructing a given continuous-time signal from its sample values. Interpolation gives either approximate or exact reconstruction or recovery of the continuous-time signal.

One very simple interpolation procedure is known as zero-order hold. Another useful interpolation procedure is called **linear interpolation**. In linear interpolation process, the adjacent samples or sample points are connected by straight lines as shown in figure 6.14.

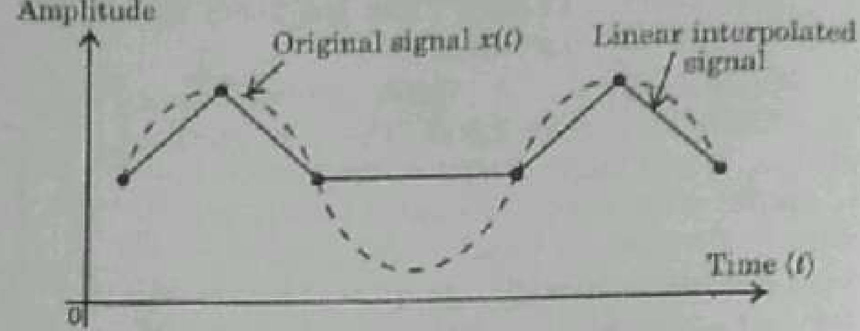

Fig. 6.14. Illustration of linear interpolation between sample points.

We may also use higher order intepolation formulae for reconstructing the continuous-time signal from its sample values. In fact, in more complicated interpolation formulae, the sample points may be connected by higher order polynomials or other mathematical functions.

However, for a band-limited continuous-time signal, if the sampling instants are sufficiently close then the signal may be reconstructed exactly by using a low pass filter. In this case, an exact interpolation can be carried out between various sample points.

Mathematical Analysis

As discussed earlier, a signal $x(t)$ band-limited to f_m Hz can be reconstructed (interpolated) completely from its samples. This is achieved by passing the sampled signal through an ideal low-pass filter of cut-off frequency f_m Hz.

The expression for sampled signal is written as

$$g(t) = x(t) \cdot \delta_{T_s}(t) \qquad \text{...(6.16)}$$

or $$g(t) = \frac{1}{T_s}[x(t) + 2\,x(t)\cos\omega_s t + 2\,x(t)\cos 2\omega_s t + ...] \qquad \text{...(6.17)}$$

From above equation, it may be observed that the sampled signal contains a component $\frac{1}{T_s} \times g(t)$.

To recover $x(t)$ or $X(j\omega)$, the sampled signal must be passed through an ideal low-pass filter of bandwidth of f_m Hz and gain T_s.

Therefore, the reconstruction or interpolating filter transfer function may be expressed as

$$H(j\omega) = T_s \times \text{rect}\left(\frac{\omega}{4\pi f_m}\right) \qquad \text{...(6.18)}$$

The impulse response $h(t)$ of this filter is the inverse Fourier transform of $H(j\omega)$.

$$h(t) = F^{-1}[H(j\omega)]$$

$$h(t) = F^{-1}\left[T_s \,\text{rect}\left(\frac{w}{4\pi f_m}\right)\right]$$

$$h(t) = 2\,f_m\,T_s\,\sin c(2\pi\,f_m t) \qquad \text{...(6.19)}$$

Assuming that sampling is done at Nyquist rate, then

$$T_s = \frac{1}{2f_m}$$

So that $\quad 2f_m T_s = 1$

Putting this value of $2f_m T_s$ in equation (6.19) we have

$$h(t) = 1 \cdot \sin c\,(2\pi f_m t)$$
$$= \sin c\,(2\pi f_m t) \qquad ...(6.20)$$

Figure 6.15(b) shows the graph of $h(t)$.

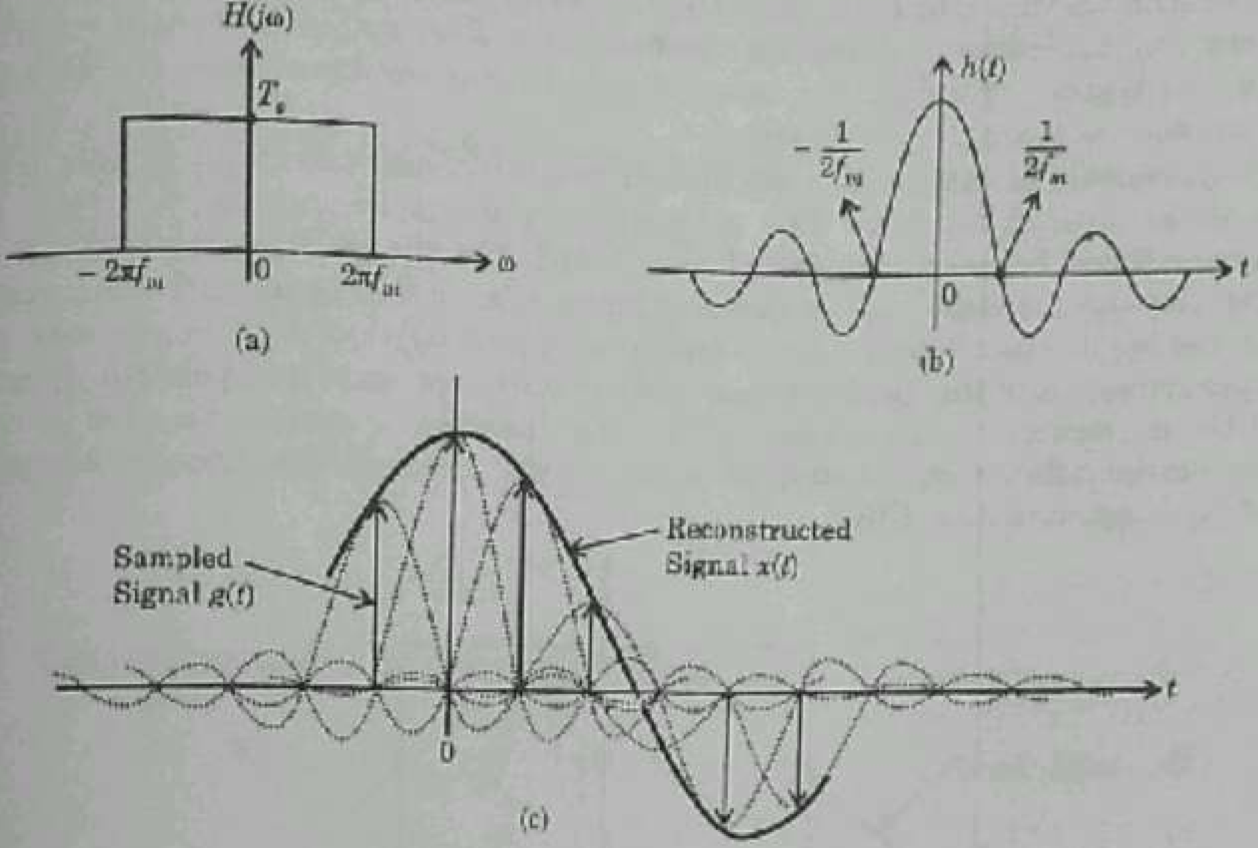

Fig. 6.15.

From figure, it may be observed that $h(t) = 0$ at all Nyquist sampling instants $t = \pm\, n/2f_m$ except at $t = 0$.

Now, when the sampled signal $g(t)$ is applied at the input of this filter, the output will be $x(t)$.

Each sample in $g(t)$, being an impulse, produces a sinc pulse of height equal to the strength of the sample.

Addition of the sin c pulses produced by all the samples results in $x(t)$.

For instant, the k^{th} sample of the input $g(t)$ is the impulse $x(kT_s)\,\delta(t - kT_s)$.

The filter output of this impulse will be $x(kT_s)\,h(t - kT_s)$.

Therefore, the filter output to $g(t)$, which is $x(t)$, may be expressed as a sum

$$x(t) = \sum_k x(kT_s)\, h(t - kT_s) \qquad ...(6.21)$$

$$= \sum_k x(kT_s) \sin c\,[2\pi f_m (t - kT_s)] \qquad ...(6.22)$$

$$x(t) = \sum_k x(kT_s) \sin c\,(2\pi f_m - k\pi) \qquad T_s = \frac{1}{2f_m} \qquad ...(6.23)$$

Equation (6.23) is known as the interpolation formula, which provides values of $x(t)$ between samples as a weighted sum of all the sample values.

In the proof of sampling theorem, it is assumed that the signal $x(t)$ is strictly band-limited. But, in general, an information signal may contain a wide range of frequencies and cannot be strictly band-limited. This means that the maximum frequency f_m in the signal $x(t)$ cannot be predictable. Therefore, it is not possible to select suitable sampling frequency f_s.

Interpolation using the impulse response $h(t)$ of an ideal low pass filter given by equation (6.23) is called **band-limited interpolation.** It implements exact reconstruction if signal $x(t)$ is band-limited and it satisfies the condition of sampling theorem. According to sampling theorem for perfect reconstruction of a band-limited signal, sampling frequency should be greater than twice the highest frequency component of the signal.

However, in many cases we always prefer simpler interpolating functions such as zero-order hold. The zero-order hold can be viewed as a form of interpolation between sample values in which the interpolating function $h(t)$ is the impulse response $h_0(t)$ as shown in figure 7.16. In this figure $x_0(t)$ corresponds to the approximation of continuous-time signal $x(t)$ and $h_0(t)$ represents an approximation of the ideal low pass filter required for exact interpolation. Figure 6.16. illustrates the magnitude of transfer function or system function of the zero-order hold interpolation filter, superimposed on the desired transfer function of each interpolation filter.

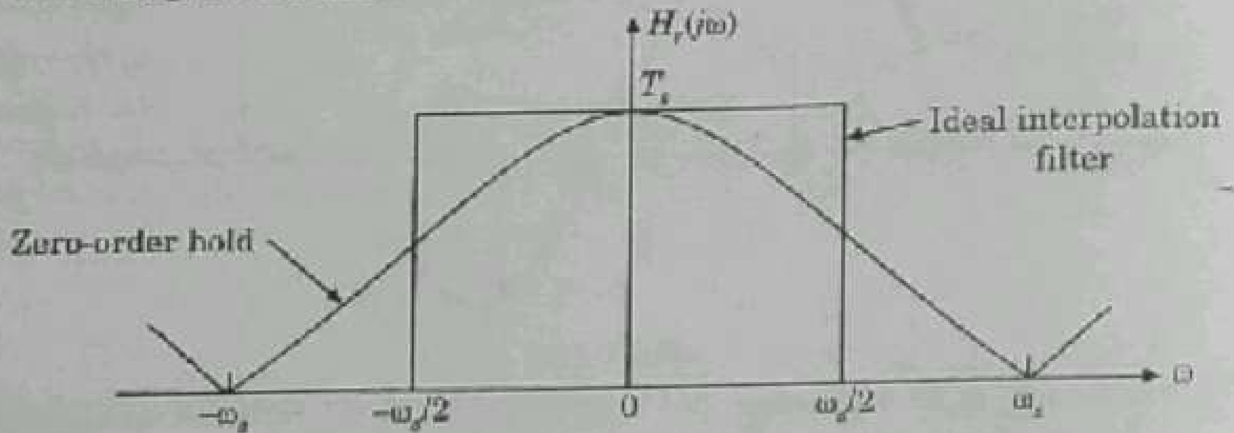

Fig. 6.16. Transfer function for the zero-order hold and for the ideal interpolation filter.

We observe from figure 6.16 and figure 6.5 that zero-order hold is a very rough approximation. For some cases, it is sufficient. If additional low pass filtering is applied in a given application, it will tend to improve the overall interpolation.

If the crude interpolation provided by the zero-order hold is not sufficient, we can use a variety of smoother interpolation methods and these are collectively known as **higher order holds.** A zero-order hold produces an output signal which is discontinuous. In contrast, a linear interpolation produces reconstruction which is continuous and its derivatives are discontinuous due to the change in slope at the sample points.

Linear interpolation is also referred to as a first-order hold linear interpolation. It is given by equation $x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, h(t - nT_s)$ with triangular impulse response shown in figure 6.17.

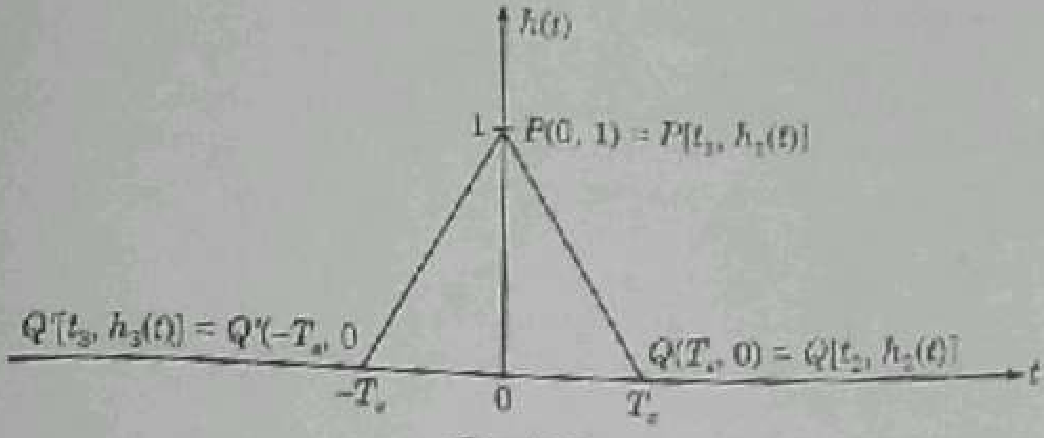

Fig. 6.17.

6.8.1. Evaluation of Transfer Function of a Filter with Triangular Impulse Response

The equation of line PQ can be determined as

$$[h(t) - h_1(t)] = \left[\frac{h_2(t) - h_1(t)}{t_2 - t_1}\right](t - t_1)$$

or $$[h(t) - 1] = \left[\frac{0-1}{T_s - 0}\right](t-0) = \frac{-t}{T_s}$$

or $$h(t) = 1 - \frac{t}{T_s}$$

The equation of line PQ' can be determined as

$$[h(t) - h_1(t)] = \left[\frac{h_3(t) - h_1(t)}{t_3 - t_1}\right](t - t_1)$$

or $$[h(t) - 1)] = \left[\frac{0-1}{-T_s - 0}\right](t-0)$$

or $$h(t) = 1 + \frac{t}{T_s} = \begin{cases} 1 - \dfrac{t}{T_s}, & \text{for } 0 \le t \le T_s \\ 1 + \dfrac{t}{T_s}, & \text{for } -T_s \le t \le 0 \end{cases}$$

- Frequency response of triangular impulse response given in above expression can be determined by taking CTFT of $h(t)$ as

$$H(j\omega) = \text{CTFT}\,[h(t)] = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

or $$H(j\omega) = \int_{-\infty}^{0}\left(1 + \frac{t}{T_s}\right)e^{-j\omega t}dt + \int_{0}^{\infty}\left(1 - \frac{t}{T_s}\right)e^{-j\omega t}dt = \frac{1}{T_s}\left[\frac{\sin(\omega T_s/2)}{\omega/2}\right]^2$$

This expression is the transfer function or frequency response of the low pass filter with triangular impulse response. The transfer function of the first order hold has been shown imposed on the transfer function of the ideal interpolation filter.

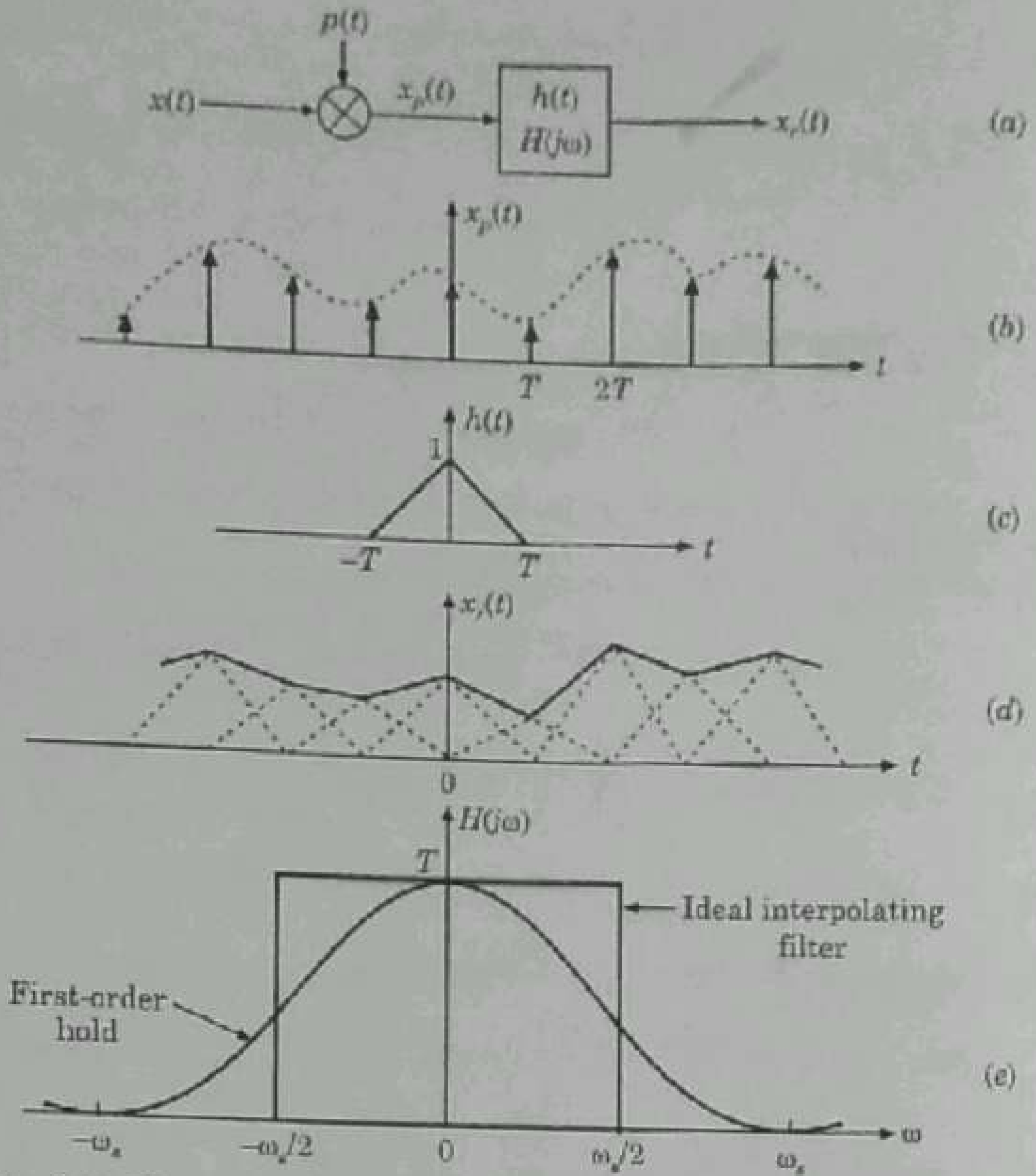

Fig. 6.18. Illustration of linear interpolation (first-order hold) as impulse train sampling is followed by convolution with a traingular impulse response.
(a) System for sampling and reconstruction of a continuous-time signal.
(b) impulse train of samples of a continuous-time signal
(c) impulse response which represents a first-order hold
(d) first-order hold is applied to the sampled signal $x_p(t)$
(e) A comparison of transfer funciton of an ideal interpolation filter and first-order hold.

6.8.2. Effect of under sampling : Aliasing

(U.P. Tech., Semester Examination, 2003-04)

When a continuous-time band-limited signal is sampled at a rate lower than Nyquist rate $f_s < 2 f_m$, then the successive cycles of the spectrum $G(jw)$ of the sampled signal $g(t)$ overlap with each other as shown in figure 6.16.

Hence, the signal is under-slampled in this case ($f_s < 2 f_m$) and some amount of aliasing is produced in this under-sampling process. In fact, aliasing is the phenomenon in which a high frequency component in the frequency-spectrum of the signal takes identity of a lower-frequency component in the spectrum of the sampled signal.

From figure 6.9 it is clear that because of the overlap due to aliasing phenomenon, it is not possible to recover original signal $x(t)$ from sampled signal $g(t)$ by low-pass filtering since the spectral components in the overlap regions add and hence the signal is distorted.

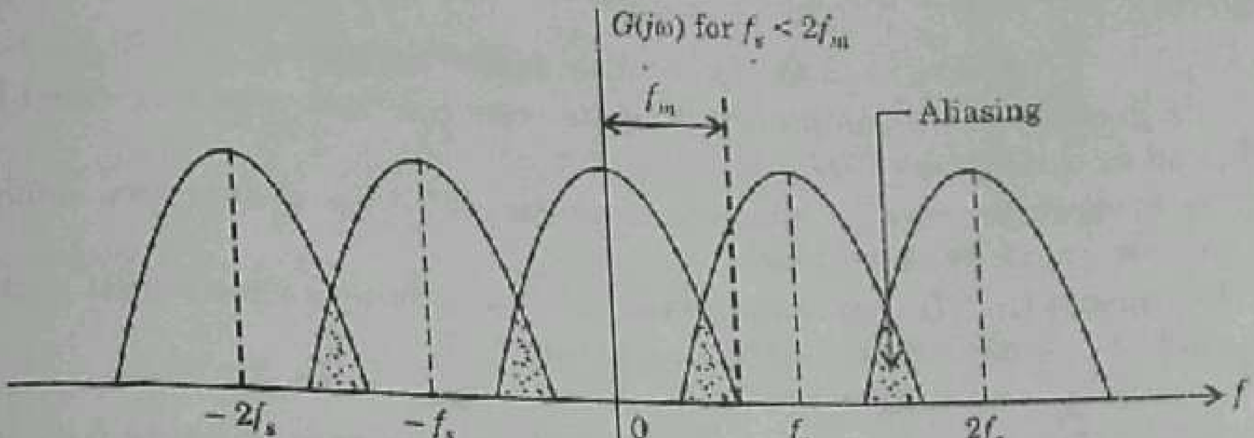

Fig. 6.19. Spectrum of the sampled signal for the case $f_s < 2f_m$

Since any information signal contains a large number of frequencies, so, to decide a sampling frequency is always a problem. Therefore, a signal is first passed through a low-pass filter. This low-pass filter blocks all the frequencies which are above $f_m Hz$. This process is known as band limiting of the original signal $x(t)$. This low-pass filter is called ***prelias filter*** because it is used to prevent aliasing effect. After band-limiting, it becomes easy to decide sampling frequency since the maximum frequency is fixed at f_m Hz.

In short, to avoid aliasing :

(i) Preaias filter must be used to limit band of frequencies of the signal to f_m Hz.

(ii) Sampling frequency 'f_s' must be selected such that

$$f_s > 2f_m \qquad \text{...(6.24)}$$