(19)

Imp. [G.B.T.U. 2013 ; M.T.U. 2013, 2014]

Poisson Distribution :→ It is a Particular limiting form of the Binomial distribution when p or q is very small and n is very big. while Poisson distribution is $$P(r) = \frac{m^r e^{-m}}{\lfloor r}$$, where m is the mean

Proof:– In Binomal distribution

$$P(r) = {}^nC_r\, q^{n-r} p^r = {}^nC_r\, (1-p)^{n-r} p^r$$

Since mean $m = np$ $\Rightarrow p = \frac{m}{n}$

$$= {}^nC_r \left[1-\frac{m}{n}\right]^{n-r} \left(\frac{m}{n}\right)^r$$

$$= \frac{n(n-1)(n-2)\cdots(n-\overline{r-1})}{\lfloor r} \left(\frac{m}{n}\right)^r \left(1-\frac{m}{n}\right)^{n-r}$$

$$= \frac{\frac{n}{n}\left(\frac{n}{n}-\frac{1}{n}\right)\left(\frac{n}{n}-\frac{2}{n}\right)\cdots\left[\frac{n}{n}-\frac{\overline{r-1}}{n}\right] m^r \left(1-\frac{m}{n}\right)^n}{\lfloor r \left(1-\frac{m}{n}\right)^r}$$

$$= \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right] m^r \left(1-\frac{m}{n}\right)^n}{\lfloor r \left(1-\frac{m}{n}\right]^r}$$

Taking limits, when n tends to infinity [Since n is very Large]

$$\lim_{n\to\infty}\left[1-\frac{m}{n}\right]^n = \lim_{n\to\infty}\left[\left(1-\frac{m}{n}\right)^{-\frac{n}{m}}\right]^{-m} = e^{-m}$$

$$\Rightarrow P(r) = \frac{m^r}{\lfloor r} e^{-m}$$ Proved.

Note 1:- m is called the Parameter of the distribution.

Note 2. The sum of the Probabilities $P(r)$ for $r = 0, 1, 2, \ldots$ is 1,

Since $P(0)+P(1)+P(2)+\cdots = e^{-m} + \frac{m e^{-m}}{\lfloor 1} + \frac{m^2 e^{-m}}{\lfloor 2} + \frac{m^3 e^{-m}}{\lfloor 3} + \cdots$

$$= e^{-m}\left[1+\frac{m}{\lfloor 1}+\frac{m^2}{\lfloor 2}+\frac{m^3}{\lfloor 3}+\cdots\right] = e^{-m}\cdot e^{m} = 1$$

Recurrence Formula for the Poission Distribution

For Poisson distribution, $P(r) = \dfrac{e^{-m} m^r}{\lfloor r}$, $P(r+1) = \dfrac{e^{-m} m^{r+1}}{\lfloor r+1}$

$$\therefore \frac{P(r+1)}{P(r)} = \frac{m^{r+1}}{m^r} \cdot \frac{\lfloor r}{\lfloor r+1} = \frac{m}{r+1}$$

or $P(r+1) = \dfrac{m}{r+1} P(r)$, $r = 0, 1, 2, 3, -----$

Mean and Variance of the Poisson distribution [A.K.T.U. 2018]

For the Poisson distribution, $P(r) = \dfrac{e^{-m} m^r}{\lfloor r}$

$$\therefore \text{Mean } \mu = \sum_{r=0}^{\infty} r P(r) = \sum_{r=0}^{\infty} r \frac{e^{-m} m^r}{\lfloor r}$$

$$= e^{-m} \sum_{r=1}^{\infty} \frac{m^r}{\lfloor r-1} = e^{-m}\left(m + \frac{m^2}{\lfloor 1} + \frac{m^3}{\lfloor 2} + ---\right)$$

$$= m e^{-m}\left[1 + \frac{m}{\lfloor 1} + \frac{m^2}{\lfloor 2} + ---\right].$$

$$= m e^{-m} \cdot e^{m} = m = \text{mean.}$$

Variance $\sigma^2 = \sum_{r=0}^{\infty} r^2 P(r) - \mu^2 = \sum_{r=0}^{\infty} r^2 \dfrac{m^r e^{-m}}{\lfloor r} - m^2$

$$= e^{-m} \sum_{r=1}^{\infty} \frac{r^2 m^r}{\lfloor r} - m^2$$

$$= e^{-m}\left[\frac{1^2 m}{\lfloor 1} + \frac{2^2 m^2}{\lfloor 2} + \frac{3^2 m^3}{\lfloor 3} + \frac{4^2 m^4}{\lfloor 4} + ---\right] - m^2$$

$$= m e^{-m}\left[1 + \frac{2m}{\lfloor 1} + \frac{3m^2}{\lfloor 2} + \frac{4m^3}{\lfloor 3} + ---\right] - m^2$$

$$= m e^{-m}\left[1 + \frac{(1+1)m}{\lfloor 1} + \frac{(1+2)m^2}{\lfloor 2} + \frac{1+3}{\lfloor 3} + ---\right] - m^2$$

$$= m e^{-m}\left[1 + \frac{m}{\lfloor 1} + \frac{m^2}{\lfloor 2} + \frac{m^3}{\lfloor 3} + ---\right] + \left[\frac{m}{\lfloor 1} + \frac{2m^2}{\lfloor 2} + \frac{3m^3}{\lfloor 3} + ---\right] - m^2$$

$$= m e^{-m}\left[e^{m} + m\left[1 + \frac{m}{\lfloor 1} + \frac{m^2}{\lfloor 2} + ---\right]\right] - m^2$$

$$= m e^{-m}\left[e^{m} + m e^{m}\right] - m^2 = m e^{-m} e^{m} (1+m) - m^2$$

$$= m(1+m) - m^2 = m$$

Hence Variance $= m$