CHAUDHARY CHARAN SINGH UNIVERSITY, MEERUT

TITLE OF E-CONTENT

INTEGRATION OF FUNCTIONS
Indefinite Integrals and Its Economic Application

Name of Content Creator:	**Dr. SANJEEV KUMAR**
Designation:	**Associate Professor**
Name of Department:	**Economics**
Name of University:	**Chaudhary Charan Singh University, Meerut**

INTEGRATION OF FUNCTIONS
Indefinite Integrals and Its Economic Application

1.0 LEARNING OUTCOMES OF THE CHAPTER

After completion of the present chapter, you should be able to;

- Describe integration by using area under curve
- *Evaluate an indefinite integral using an anti-derivative*
- *Describe an indefinite integral and its economic application*

1.1 AREA UNDER CURVE

Introduction

There are two limiting processes of Calculus. First one is differentiation in which we study about the tangent to the curve or rate of change in one variable due to change in other variables. On the other hand, second one is integration, in which we study about the area under curve integration can be defined as:

"Integration is the process of finding the function from it's derivative and this function is called the integral of the function".

Basically, we use integration to find out area under a curve. We can also find the area under curve by geometrically. However, concept of integration and differentiation do not depend on geometry as analytically. A geometrical interpretation is used only to understand intuitively.

Let $y = f(x)$ be a continuous and positive function on the closed interval [a, b] in the figure (1). We have to find the area of given function on the closed interval [a, b]. Now the question is how do we compute area (A) under the given graph.

Further, suppose A(x) is the area that measures the area under curve $y = f(x)$ on the closed interval [a, x]

It is clear from the given figure (1) that;

$$A(a) = 0$$

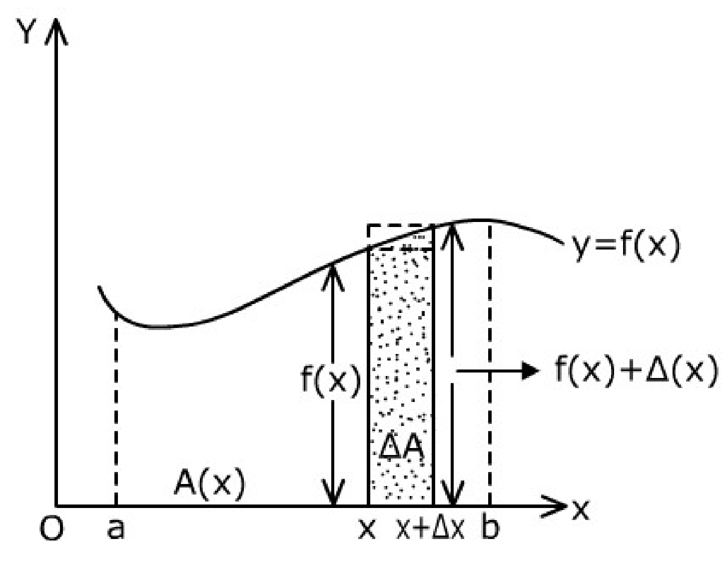

Figure 1

Because, there is no area from 'a' to 'a' and the total area can be defined as,

A= A(b)

Now, we suppose that 'x' increases by Δx amount. Then, $A(x+\Delta x)$ is the area under curve $y = f(x)$ over the closed interval $[a, x+\Delta x]$, Hence, the required area is given by;

$$A(x+\Delta x) - A(x)$$

It is the area {ΔA} under the curve $y = f(x)$ over the closed interval $[x, x+\Delta x]$. Let, Δ A be very small i.e. magnified and this area can not be exceed the area of rectangle with edges Δx and $f(x+\Delta x)$ and cannot be lesser than area of the rectangle with edges Δx and f(x). Hence, $\forall x > 0$, then;

$$f(x)\Delta x \le A(x+\Delta x) - A(x) \le f(x+\Delta x)\Delta x$$

$$OR, f(x) \le \frac{A(x+\Delta x) - A(x)}{\Delta x} \le f(x+\Delta x)$$

If we take $\Delta x \to 0$ in the above equation then the interval $[x, x+\Delta x]$ shrinks to the single point 'x' and the value $f(x+\Delta x)$ approaches $f(x)$. So, the function A (x) is differentiable and it measures the area under the curve $y = f(x)$ over the closed interval $[a, x]$. Then, the derivative of the function is given by;

$$A'(x) = f(x) \quad \{\forall x \in (a,b)\}$$

This proves that the derivatives of the area function A (x) is a curve height function {i.e. y = f (x)}

Now, suppose $F(x)$ is another continuous function with the function $y = f(x)$ as its derivative;

$$\text{Then,} \quad F'(x) = A'(x) = f(x) \quad \forall x \in (a,b)$$

$$\text{Because,} \quad \frac{d}{dx}\left[A(x) - F(x)\right] = A'(x) - F'(x) = 0$$

It must also be true that,

$$A(x) = F(x) + C \quad \text{\{C is some constant\}}$$

If A(a) = 0, then

A (a) = F(a) +C = 0

Or C = -F(a), put this value in above equation

$$\boxed{\begin{aligned} &A(x) = F(x) + C = F(x) - F(a)\{when, F'(x) = f(x)\} \\ &At, x = b, then, A(x) = F(b) - F(a) \end{aligned}}$$

In short, the method for finding the area under the curve $y = f(x)$ and its domain (a,b) or above the x –axis from $x = a$ to $x = b$ has following steps;

- Find an arbitrary function F(x),that is continuous over the interval (a, b) such that

$$F'(x) = f(x) \qquad \forall x \in (a,b) \text{ ------------------- (i)}$$

- Then the required area of the function is given by

$$A(x) = F(b) - F(a) \qquad \text{-------------------------------(ii)}$$

What happens, if the function $y = f(x)$ has negative value in [a, b]. At this condition, the required area is $A(x) = -[f(b) - F(a)]$. Further, we know that, the area of a region is always positive. So $A(x)$ is also positive.

Example 1:

Find the area under the straight line $y = f(x) = x$ over the interval [0,1]

Solution:

We have to find the shaded area (A) in the given figure. According to above equation (i) and (ii) given above, we must find a function, that has x as its derivative.

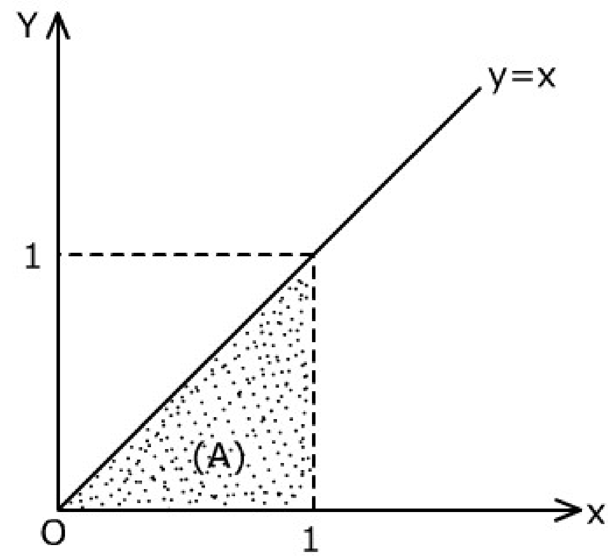

Figure 2

Then,

$$F(x) = \frac{x^2}{2}$$

$$\left\{\because \frac{d}{dx}(ax^n) = anx^{n-1} = x, here, n = 2 \; \& \; a = \frac{1}{2}\right\}$$

$$F'(x) = 2\frac{x}{2} = x$$

Thus, the required area is given by;

$$A = F(1) - F(0)$$

$$= \frac{1}{2} - 0 = \frac{1}{2}$$, This answer is reasonable.

Example 2:

Compute the area under the parabola; $y = f(x) = x^2$ over the interval [a, b]

Solution:

We have estimated the shaded area A in the given figure (3). According to equation (i) and (ii) given above, we have to find a function, that has x as its derivative.

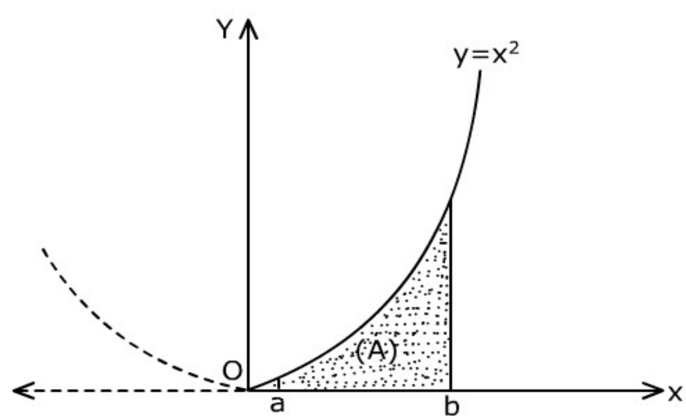

Figure 3

Let,

$$F(x)=\frac{1}{3}x^3$$

$$Then, F'(x)=f(x)=\frac{1}{3}\times 3x^2=x^2$$

Thus, the required area is given by

$$A=F(b)-F(a)=\frac{1}{3}b^3-\frac{1}{3}a^3$$

$$A=\frac{1}{3}\left[b^3-a^3\right]$$

Example 3:

Compute the area 'A' under the straight line $y=f(x)=ax+b$ over the interval $[\alpha,\beta]$

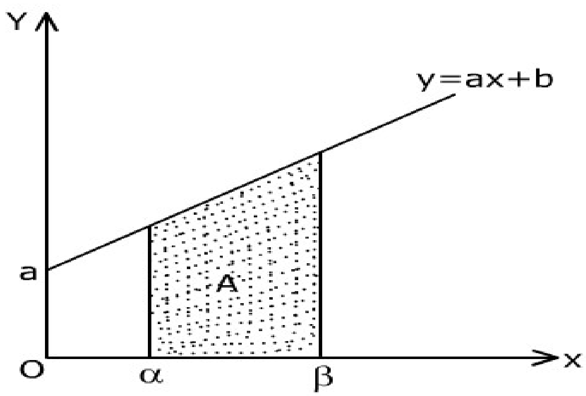

Figure 4

Solution:

Let, the shaded area under the straight line be given by 'A', then from equations (i) and (ii) given above, we get;

$$F(x)=\frac{1}{2}ax^2+bx\left\{\because\frac{d}{dx}(ax^n)=anx^{n-1}=ax+b\right\}$$

$$\text{Then, } F'(x)=\frac{1}{2}.2ax+b.1$$

$$= ax+b$$

So, the required area A is given by

$$A=F(b)-F(a)$$

$$A=F(\beta)-F(\alpha)$$

$$=\frac{1}{2}a\beta^2+b\beta-\frac{1}{2}a\alpha^2-b\alpha$$

$$=\frac{1}{2}a(\beta^2-\alpha^2)+b(\beta-\alpha)$$

$$=(\beta-\alpha)\left\{\frac{a(\beta+\alpha)+2b}{2}\right\}$$

Example 4: Find the shaded area 'A' of the function $y=f(x)=e^{x/3}-3$ over the closed interval [0, 3 ln 3]

Solution: First we have to find the function $F(x),$ whose derivative is $e^{x/3}-3$

By using the results of equation (i) and (ii) given above, we take the function,

$$F(x)=3e^{x/3}-3x \qquad \left\{\because\frac{d}{dx}(e^x)=e^x\right\}$$

Then, $F'(x) = f(x) = e^{x/3} - 3$

So, the required area A is given by,

$$A = -[F(b) - F(a)]$$
$$= -(3e^{\ln 3} - 3\times 3\ln 3 - 3e^{\circ}$$
$$= -(9-9\ln 3-3) = 9\ln 3-6 \qquad \text{(ignore –ve sign)}$$
$$\therefore A = 3.89 \text{ units}$$

1.2 INDEFINITE INTEGRALS

Introduction

The previous section of the present chapter discusses the problem of finding an anti-derivative of the function $f(x)$ i.e. a function $F(x)$ whose derivative is $f(x)$.

$$F'(x) = f(x)$$

Anti-derivative is an appropriate name. Usually in practice, we call $F(x)$ an indefinite integral of $f(x)$. It is denoted by the symbol $\int$.

"If $f(x)$ is the differential coefficient of function $F(x)$, then $F(x)$ is the integral of $f(x)$"

By symbolically, if $\dfrac{d}{dx}[F(x)] = f(x)$

Then, $\boxed{\int f(x)dx = F(x) + C}$

Here 'C' is the constant term. We know that differentiation of constant term is zero. If integral constant 'C' can take any value then the integral is called indefinite integral.

Basic Rule of Integration

Power Rule: It is defined as;

$$\boxed{\int x^n dx = \frac{1}{n+1}x^{n+1} + C} \qquad \{ n \neq -1 \}$$

Example: $\int x\, dx = \dfrac{x^2}{2} + C$

Exponential Rule: It is defined as;

$$\boxed{\int e^x dx = e^x + C}$$

And, $\boxed{\int a^x dx = \dfrac{a^x}{\log_e a} + C}$ $\qquad \{ a > 0 \,\&\, a \neq 1 \}$

Examples:

$$\int e^{-x}dx = e^{-x} + C$$

$$\int e^{ax}dx = \frac{1}{a}e^{ax} + C \qquad \{a \neq 0\}$$

$$\int 2^{x}dx = \frac{2^{x}}{\log_e 2} + C$$

Logarithmic Rule: It is defined as;

$$\boxed{\int \frac{1}{x} = \ln|x| + C}$$

Example:

$$\int \frac{1}{t}dt = \ln|t| + C$$

Some standard Results of Integration

- **Constant multiple property**

$$\int af(x)dx = a\int f(x)dx \qquad \{a \text{ is the real constant}\}$$

- **Integral of sum**

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

In general,

$$\boxed{\int [a_1 f_1(x) + a_2 f_2(x) + ... + a_n fn(x)]\, dx = a_1 \int f_1(x)dx + a_2 \int f_2(x)dx + ... + a_n \int fn(x)dx}$$

- **Integral of Difference**

$$\int [F(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$$

In general,

$$\boxed{\int [a_1 f(x) - a_2 f(x) - ...... - a_n fn(x)]dx = a_1 \int f_1(x)dx - a_2 \int f_2(x)dx - .... - a_n \int fn(x)dx}$$

Integral of Multiplication

$$\boxed{\int f(x)g(x)dx = f(x)\int g(x)dx - \int [\ \frac{d}{dx} f(x) \ \int \mathrm{g(x)dx}]dx}$$

This property is also known as integration by part.

SOME OTHER RESULTS

$$\boxed{\int \frac{1}{x^2+a^2}dx = \log\left[x+\sqrt{x^2+a^2}\right]+C}$$

$$\boxed{\int \frac{1}{x^2-a^2}dx = \log\left[x+\sqrt{x^2-a^2}\right]+C}$$

$$\boxed{\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left[\frac{(a+x)}{(a-x)}\right]+C}$$

Example 1:

Find the integral $\int (5x^4+3x^2+2x-1)dx$

Solution:

$$\int (5x^4+3x^2+2x-1)dx$$
$$= \int 5x^4dx + \int 3x^2dx + \int 2xdx - \int dx$$
$$= 5\int x^4dx + 3\int x^2dx + 2\int xdx - \int dx$$
$$= 5\frac{x^5}{5}+C_1+\frac{3.x^3}{3}+C_2+2\frac{x^2}{2}+C_3-x+C_4$$
$$= x^5+x^3+x^2-x+C_1+C_2+C_3+C_4$$
$$= x^5+x^3+x^2-x+C \qquad \{C = C_1+C_2+C_3+C_4\}$$

Example 2: Evaluate $\int (e^x+\frac{1}{x^3}+1)dx$

Solution:

$$\int (e^x+\frac{1}{x^3}+1)dx$$
$$= \int e^xdx + \int x^{-3}dx + \int 1dx$$
$$= e^x - \frac{1}{2}x^{-2}+x+c$$
$$= e^x - \frac{1}{2x^2}+x+c$$

Example 3: Find the integral $\int \frac{(x+1)^2+2x^{-1/2}}{\sqrt{x}}dx$

Solution:

$$\int\left[\frac{(x+1)^2+2x^{-1/2}}{\sqrt{x}}\right]dx$$
$$= \int\left[\frac{x^2+2x+1+2x^{-1/2}}{x^{1/2}}\right]dx$$

$$= \int (x^{3/2} + 2x^{1/2} + x^{-1/2} + 2\frac{1}{x})dx$$

$$= \frac{2}{5}x^{5/2} + \frac{4}{3}x^{3/2} + 2x^{1/2} + 2\ln x + c$$

Example 4: Compute $\int \frac{x^2}{x+1}dx$

Solution: Let, $\int \frac{x^2}{x+1}dx$

$$= \int \left\{ \frac{x^2 - 1 + 1}{x+1} \right\} dx$$

$$= \int \frac{(x-1)(x+1)+1}{(x+1)}dx$$

$$= \int (x-1)dx + \int \frac{1}{(x+1)}dx$$

$$= \frac{x^2}{2} - x + \log|x+1| + c$$

Example5: Evaluate $\int \frac{dx}{\sqrt{x+c} - \sqrt{x-d}}$

Solution: Let, $\int \frac{dx}{\sqrt{x+c} - \sqrt{x-d}}$

$$= \frac{\sqrt{x+c} + \sqrt{x-d}\,)}{(\sqrt{x+c} - \sqrt{x-d}\,)(\sqrt{x+c+\sqrt{x-d}}}dx$$

$$= \int \frac{\sqrt{x+c} + \sqrt{x-d}\,)}{(x+c)-(x-d)}dx$$

$$= \frac{1}{(c+d)}\int (x+c)^{1/2}dx + \frac{1}{(c+d)}\int (x-d)^{1/2}dx$$

$$= \frac{1}{(c+d)} \quad \frac{2}{3}(x+c)^{3/2} + \frac{1}{c+d} \quad \frac{2}{3}(x-d)^{3/2} + c$$

$$= \frac{2}{3}\frac{1}{(c+d)} \quad \left[(x+c)^{3/2} + (x-d)^{3/2}\right] + c$$

Example 6: Find the integration $\int (6x+9)^8 dx$

Solution: By using substitution method,

Let $y = 6x + 9$

Then, $dy = 6dx \quad or \quad dx = \frac{1}{6}dy$

So, we get, $$\int (6x+9)^8 dx = \frac{1}{6}\int y^8 dy$$

$$= \frac{1}{6} \quad \frac{y^9}{9} + c$$

Now putting the value; $y = 6x + 9,$ then

$$\int (6x+9)^8 dx = \frac{1}{54}(6x+9)^9 + C$$

Example 7: Evaluate $\int \frac{-x^2}{4-x^2} dx$

Solution: Let, $\int \frac{-x^2}{x^2-4} dx \quad = \quad \int \frac{-x^2+4-4}{4-x^2} dx$

$$= \int 1 dx - 4\int \frac{1}{4-x^2} dx$$

$$= x - 4.\frac{1}{2\times 2}\log\left[\frac{2+x}{2-x}\right] + c \quad \text{(by the formulae)}$$

$$= x - log\left[\frac{2+x}{2-x}\right] + c$$

Example 8: Evaluate $\int x^2 e^{2x}$

Solution: Let, $I = \int x^2 e^{2x} dx$

By using the formulae for integration by part,

$$I = x^2 \int e^{2x} dx - \int \left\{ \frac{d}{dx} x^2 \int e^{2x} dx \right\} dx$$

$$= x^2 \frac{e^{2x}}{2} - \int \frac{2x.e^{2x}}{2} dx$$

$$= \frac{1}{2} x^2 e^{2x} - \int x e^{2x} dx$$

$$= \frac{1}{2} x^2 e^{2x} - \left[x \int e^{2x} dx - \int \left\{ \frac{d}{dx}.x \int e^{2x} dx \right\} \right] dx$$

$$= \frac{1}{2} x^2 e^{2x} - \left[\frac{x e^{2x}}{2} - \int 1.\frac{e^{2x}}{2} dx \right]$$

$$= \frac{1}{2} x^2 e^{2x} - \left[\frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} \right] + c$$

$$= \frac{1}{2} x^2 e^{2x} - \frac{1}{2} x e^{2x} + \frac{1}{4} e^{2x} + c$$

$$= \frac{1}{2} e^{2x} \left[x^2 - x + \frac{1}{2} \right] + c$$

Example 9: Find $\int \frac{1}{4x^2-9}dx$

Solution: $\int \frac{1}{4x^2-9}dx = \int \frac{1}{(2x+3)(2x-3)}dx$

Now, let; $\frac{1}{(2x+3)(2x-3)} = \frac{A}{(2x-3)} + \frac{B}{(2x+3)}$................$(i)$

$$= \frac{2Ax+3A+2Bx-3B}{4x^2-9}$$

$$= \frac{2x(A+B)+3(A-B)}{4x^2-9}$$

Now compare both sides of the equation; $2x(A+B)+3(a-B)=1$

Hence $2(A+B)=0$ or A=-B and 3(A-B)=1 or B=-1/6 and A=1/6, now by equation (i)

$$\therefore \int \frac{1}{4x^{2-9}}dx = \frac{1}{6}\int \frac{dx}{2x-3} - \frac{1}{6}\int \frac{dx}{2x+3}$$

$$= \frac{1}{12}\ln|2x-3| - \frac{1}{12}\ln|2x+3| + C$$

ECONOMIC APPLICATION OF INDEFINITE INTEGRATION

Introduction

Indefinite integration has an important role in economics. It tells about that how to calculate total economic aspect from the marginal aspects. The following section shows the role of indefinite integration in economics by illustrating some important examples.

Economic Application of Integration

There are several economic applications of indefinite integration. Some results are given below;

- **Total cost (TC)**

  $$TC = \int MC(Q)dQ$$

  Here; MC $\Rightarrow$ Marginal cost, Q $\Rightarrow$ output

- **Total Revenue (TR)**

  TR= $\int MR(Q)dQ$, Here, MR $\Rightarrow$ Marginal Revenue

- **Total Product (TP)**

TR= $\int MP(Q)dQ$,   Here, MP $\Rightarrow$ Marginal Product

Example 10: Find total cost function from the following marginal cost (MC) function;

$$MC = f'(q) = 2 + 3q^{1/2} + 5/q^{-1/2},$$

Given; f(1) = 11

Solution: $\therefore F(q) = \int f'(q)dq = \int (2 + 3q^{1/2} + 5q^{-1/2})dq = 2q + 3.\frac{q^{3/2}}{3/2} + 5.\frac{q^{1/2}}{1/2} + c$

$\therefore TC = 2q + 2q^{3/2} + c$

$\therefore$ When q=1 then 11 = 2 + 2 + c, Then, c=7

$\therefore$ Total cost function F(q) = 22 + 3$q^{3/2}$ + 10 $q^{1/2}$+ 7

Example 11: If the marginal revenue function is given; $P_m=\left\{\frac{\alpha\beta}{(x+\beta)^2} - \gamma\right\}$,

Then, show that P=$\left\{\frac{\alpha\beta}{(x+\beta)} - \gamma\right\}$ is the demand law

Solution: $\because R = P.x$ and MR=$\frac{dR}{dx}$

$$\therefore R = \int MR.dx = \int \left(\frac{\alpha\beta}{(x+\beta)^2} - \gamma\right)dx$$

$$\therefore R = \alpha\beta\frac{(x+\beta)^{-1}}{-1} - \gamma x + A$$

$$\therefore R = P.x = -\frac{\alpha\beta}{(x+\beta)} - \gamma x + A,$$

We know that if output x=0 then revenue is also zero. Then A = α

$$\therefore R = P \times x = -\frac{\alpha\beta}{(x+\beta)} - \gamma x + \alpha = \frac{\alpha x}{(x+\beta)} - \gamma x$$

OR, $P = \frac{\alpha}{x+\beta} - \gamma$,   Hence proved.

Example 12: If marginal revenue (MR) = $16-q^2$, find the maximum total revenue, also find the total, average revenue demand.

Solution: When TR is maximum, then MR= 0

$$\therefore 16-q^2=0 \Rightarrow q=\pm 4$$

$$\therefore TR=\int_0^4 MR \text{ dq}=\int_0^4 (16-q^2)dq=\left[16q-\frac{q^3}{3}\right]_0^4=\frac{128}{3}$$

Total Revenue (TR) = $\int (16-x^2)dx = 16x-\frac{x^3}{3}+c$ when $x=0$ then $c=0$

Average Revenue (AR) = $\frac{TR}{q}=16-\frac{q^2}{3}$

Then, Demand (AR) =P = $16-q^2/3$

Example 13: If marginal propensity to consume (MPC) function is given as follows; $\frac{dc}{dy}=0.5-0.001y$, then find total consumption function. Given at income zero, c is 0.02.

Solution: $\therefore C=\int \frac{dc}{dy}.dy=\int (0.5-0.001y = 0.5y-\frac{0.001}{2}y^2+A$

At $\therefore y=0$, then, C = 0.2, Hence, A = 0.2

$$\therefore C=0.5y-0.0005y^2+0.2$$

Example 14: Calculate Q(L), where Q'(L) = $6L^{1/3}$ and Q(0) = 0

Solution: Q(L) = $\int 6L^{1/3}dL=\frac{18}{4}L^{4/3}+c$

Given L=0, then Q(0)= 0+C or C=0

Then; Q(L) = $18/4L^{4/3}$

PROBLEM SET

1. Find the area under the graph of polynomial $y = f(x) = x^3$ over the interval [0,1]

2. Find the bounded area of the graph of function $y = f(x) = \frac{1}{2}(e^x + e^{-x})$ over the interval (-1,1)

3. Find the area under straight line, $y = f(x) = cx + d$ over the interval [o,1]

4. Compute the area under the parabola $y = 4x^2$ over the interval [o,1]

5. Find the integrals of the following:

   (i) $\int (4x^3 - 9x^2 + 2x + 2)dx$ (ii) $\int \frac{A}{r^{5/2}} dr$

   (iii) $\int (3t^2 + 2t - e^t)dt$ (iv) $\int x\sqrt{x} \ dx$

6. Prove that, $\int (ax+b)^\alpha dx = \frac{1}{a(\alpha+1)}(ax+b)^{\alpha+1} + c$

7. Find the integration (i) $\int \frac{1}{\sqrt{x+2}} dx$ (ii) $\int \frac{x}{2x^2+3} dx$

8. Calculate (i) $\int x\sqrt{x^2+1} \ dx, x > 0$ (ii) $\int x\sqrt[3]{x-2} \ dx$

9. If the marginal cost of producing x units for a manufacture product is MC=C'=2x+4 then find total cost function C(x). Given, fixed cost = 40

10. Evaluate $\int \frac{1}{2}(e^x + e^{-x})dx$

11. Given, $f''(t) = 1/t^2 + t^3 + 2 \ \forall t > 0 \text{ and } f(1) = 0, f'(1) = 1/4$ then find f(t).

12. Prove that,

$$\int t\sqrt{at+b}.dt = \frac{2}{15a^2}(3at-2b)(at+b)^{3/2} + c$$

13. Find the integration (i) $\int \log x \ dx$ (ii) $\int x^5 e^x dx$

14. Find the general form of the function f(x), whose third derivative is x and also given

$$f''(0) = f'(0) = f(0) = 0$$

15. Evaluate, (i) $\int \frac{1}{x^2 - a^2} dx$ (ii) $\int \frac{2x+1}{(x+1)(x-2)(x-3)}$

ANSWERS OF PROBLEM SET

1. Area (A) = $\frac{1}{4}$

2. Area (A) = $\left(e - \frac{1}{e}\right)$

3. Area (A)= $\frac{1}{2}(a+b)$

4. Area (A)= $\frac{4}{3}$

5. (i) $x^4 - 3x^3 + x^2 + 2x + c$

(ii) $-\frac{2A}{3r^{3/2}} + c$

(iii) $t^3 + t^2 - e^t + c$

(iv) $\frac{2}{5}x^{5/2} + c$

7. (i) $2[\sqrt{x} - 2\ln(\sqrt{x} + 2] + C$

(ii) $\frac{1}{4}\ln(2x^2 + 3] + C$

8. (i) $\frac{1}{3}\ln(x^2 + 1)^{3/2} + C$

(iv) $\frac{3}{14}(x-2)^{4/3}(2x+3) + c$

9. $C(x) = x^2 + 4x + 40$

10. $\frac{1}{2}(e^x - e^{-x})$

11. $t + \frac{1}{20}t^5 - \log|t|$

13. (i) $x\log x - x + c$

(ii) $x^5e^x - 5x^4 5e^x + 20x^3e^x - 60x^2e^x + 120xe^x + 120e^x + c$

14. $\frac{1}{24}x^4$

15. (i) $\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$ (ii) $-\frac{1}{12(x+a)} - \frac{3}{5(x-2)} + \frac{7}{4(x-3)}$

Additional Readings

- Allen, R.G.D, *Mathematical Analysis for Economists*, London: Macmillan and Co. Ltd

- Chiang, Alpha C., *Fundamental Methods of Mathematical Economics*, New York: McGraw Hill

- Carl P. Simon and Lawrence Blume, Mathematics for Economists, London: W .W. Norton & Co.

- Knut Sydsaeter and Peter J. Hammond, *Mathematics for Economic Analysis*, Prentice Hall

- Michael Hoy, John Livernois, Chris Mckenna, Ray Rees, Thantsis Stengos, *Mathematics for Economists*, Addison-Wesley Publishers Ltd.