TITLE OF E-CONTENT

DETERMINANTS AND MATRIX INVERSION

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DETERMINANTS AND MATRIX INVERSION

Learning Outcomes of the Present E Content

In the present chapter you will learn about the following aspects;

- Understand the concept of determinants and its order
- You will be able to apply determinant to solve the mathematical equations.
- Get to know the various properties of determinants and its types.
- Explain the Sarrus' method of inverse
- Understand the concept of inverse and its application
- Know the steps and tools to calculate and solve determinants.
- Understand the different types of determinants and its various concepts.
- Solve the simultaneous equations using Cramer's Rule.
- Describe the trivial or nontrivial solution of homogeneous system of equations

Introduction

The present chapter is developed to understand the concept of determinants and its application to find out matrix inversion. Basically, it is a part of linear algebra which eases the difficulty level of the simultaneous equations in algebra by providing means for their presentation and solution.

Determinant: Definition and Its Order

Every square matrix $A_{n\times n}$is associated with a unique number called the determinants of the matrix. If A = (a_{ij}) be an n×n matrix, then the determinant of A is denoted by |A| or det(A) or

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}.$$

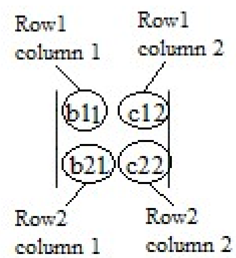
If A = (a_{11}) be an 1×1 matrix, then |A| = a_{11} i.e., the determinant is equal to the element itself.

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a 2×2 matrix then,

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \times a_{22} - a_{12} \times a_{21}$$

It is called **determinant of order two or second order determinant.** Thevalueof determinant of order two is equal to$a_{11} \times a_{22}$- $a_{12} \times a_{21}$. Here, the elements of determinants are a_{11}, a_{22}, a_{12} and a_{21}.

Thus we may represent the determinant $\begin{vmatrix} b1 & c1 \\ b2 & c2 \end{vmatrix}$ in terms of rows and columns as:



LEADING TERM: The diagonal elements in the determinant i.e. b_{11} and c_{22} are the leading term and it always has a positive sign.

Note: A determinant of the second order has two diagonal elements having positive signs and 2! = 2 terms in its expansion out of which one is positive and other is negative.

The IIIrd order determinant or determinant of order 3 can be defined as;

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$
$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

So, we have expressed the determinant of order 3 in terms of determinants of order 2. We can similarly express the determinants of higher orders.

Note: *We can expand a determinant by any row or column and it will generate the same value of the determinant every time.*

If we consider the following equation with 'k' number of equations and 'm' number of unknowns:

$$b_{11}x_1 + b_{12}x_2 + \ldots + b_{1m}x_m = c_1$$
$$b_{21}x_1 + b_{22}x_2 + \ldots + b_{2m}x_m = c_2$$
$$\begin{matrix} . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{matrix}$$
$$b_{k1}x_1 + b_{k2}x_2 + \ldots + b_{km}x_m = c_k$$

We can express this equation in a compact way and solve it by using matrices. Let

$$B = \begin{pmatrix} b11 & \cdots & b1m \\ \vdots & \ddots & \vdots \\ bk1 & \cdots & bkm \end{pmatrix}, X = \begin{pmatrix} x1 \\ \vdots \\ xm \end{pmatrix}, C = \begin{pmatrix} c1 \\ \vdots \\ ck \end{pmatrix}$$

In simple terms, it can be written as BX = C and thus be solved. This square matrix we know is non-singular and in this chapter, we use determinants to determine whether a given square matrix is non-singular/invertible or not. For a matrix to be non-singular, its determinant value should not be equal to zero.

Example: Compute the value and cofactors of the given below determinant.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 2 & 3 \\ 7 & 1 & 3 \end{vmatrix}$$

Solution: Expanding the determinant from 1st row we get,

$$|A| = 1 \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} - 2 \begin{vmatrix} 4 & 3 \\ 7 & 3 \end{vmatrix} + 3 \begin{vmatrix} 4 & 2 \\ 7 & 1 \end{vmatrix}$$

$$= 1(6-3) -2(12-21) +3(4-14)$$

$$= 3 - (-18) + (-30) = -9$$

We can expand the determinant by any row or column, if we expand it by 1st column,

$$|A| = 1 \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} - 4 \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} + 7 \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix}$$

$= 1(6-3) – 4(6-3) + 7(6-6)$

$= (3) – (12) + (0) = -9$

Cofactors of determinant;

Cofactor of element in 1st row and 1st column: $(-1)^{1+1}\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = 3$

Cofactor of element in 1st row and 2nd column: $(-1)^{1+2}\begin{vmatrix} 4 & 3 \\ 7 & 3 \end{vmatrix} = - \begin{vmatrix} 4 & 3 \\ 7 & 3 \end{vmatrix} = 9$

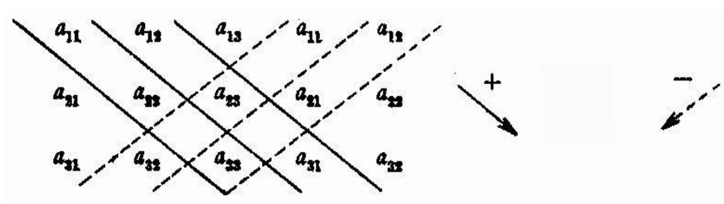
Cofactor of element in 1st row and 3rd column: $(-1)^{1+3}\begin{vmatrix} 4 & 2 \\ 7 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 7 & 1 \end{vmatrix} = -10$

And so on for other elements.

Sarrus' Rule

Sarrus' rule is the alternative way to compute determinants of order 3. This method is very convenient for many people. In this method, we write down the determinant twice, except that the second time the last column of the IInd determinant should be omitted. It is given below;

Let A= $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then



Firstly, multiple along three lines falling to the right, giving all these products a plus sign;

$a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}$ --(A)

Secondly, multiple along three lines falling to the right, giving all these products a minus sign;

$-a_{31}a_{22}a_{13}-a_{32}a_{23}a_{11}-a_{33}a_{21}a_{12}$ --(B)

The sum of equation (A) and (B) is exactly equal to determinant Ai.e$\lvert\ \rvert$,|A|.

Basic Rules for Determinant

- If all the elements of any row or column are zero, the value of the determinant is also zero, then, | A| = 0
- If we exchange all the rows of a determinant from columns and vice-versa, the determinant remains unchanged in value and signs i.e. the value of a determinant and its transpose remains same i.e.$|A| = |A|^T$
- If we interchange any two rows or two columns of a determinant, its value remains unchanged numerically but changes in sign.
- If a constant ‘c’ is multiplied (or divided) by all the elements of any one row (or column) of a determinant, then the value of the determinant is also multiplied (or divided) by ‘c’.
- If the elements of one row (or column) are identical/equal/proportional to the elements of a second row (or column), the determinant takes the value zero.
- The determinant of the product of two n×n matrices A and B is the product of the determinants of each of the factors;

$$|AB| = |A|.|B|$$

- If A be ann×n matrix and α be an real number, then;

$$|\alpha A| = \alpha^n |A|$$

- An orthogonal matrix must have determinant 1 or -1.
- If A^2 = I, then a square matrix of order n is called Involutive and its determinant is always 1 or -1.

Multiplication of Determinants

When we multiply two determinants of same order, the resultant determinant is a determinant of same order. It is given by;

$$\begin{vmatrix} b1 & c1 \\ b2 & c2 \end{vmatrix} \times \begin{vmatrix} a1 & d1 \\ a2 & d2 \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} b1 & c1 \\ b2 & c2 \end{vmatrix} \cdot \begin{vmatrix} a1 & d1 \\ a2 & d2 \end{vmatrix} & \begin{vmatrix} b1 & c1 \\ b2 & c2 \end{vmatrix} \cdot \begin{vmatrix} a1 & d1 \\ a2 & d2 \end{vmatrix} \\ \begin{vmatrix} b1 & c1 \\ b2 & c2 \end{vmatrix} \cdot \begin{vmatrix} a1 & d1 \\ a2 & d2 \end{vmatrix} & \begin{vmatrix} b1 & c1 \\ b2 & c2 \end{vmatrix} \cdot \begin{vmatrix} a1 & d1 \\ a2 & d2 \end{vmatrix} \end{vmatrix}$$

$$=\begin{vmatrix} b1a1 + c1a2 & b1d1 + c1d2 \\ b2a1 + c2a2 & b2d1 + c2d2 \end{vmatrix}$$

Example: Let, $\begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} \times \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = \begin{vmatrix} 4\times2+5\times3 & 4\times1+5\times2 \\ 1\times2+2\times3 & 1\times1+2\times2 \end{vmatrix}$

$$= \begin{vmatrix} 8+15 & 4+10 \\ 2+6 & 1+4 \end{vmatrix} = \begin{vmatrix} 23 & 14 \\ 8 & 5 \end{vmatrix}$$

Ad-joint or Adjugate and Cofactors of a Determinant

Suppose we have a determinant |A| and its ad-joint is represented as|A|' or Adj A. It is also known as augmented matrix. The elements in |A|'are the cofactors of the corresponding elements of |A| , i.e.,

$$\text{Let } |A| = \begin{vmatrix} b1 & c1 & d1 \\ b2 & c2 & d2 \\ b3 & c3 & d3 \end{vmatrix}, \text{ then } |A'| = \begin{vmatrix} B1 & C1 & D1 \\ B2 & C2 & D2 \\ B3 & C3 & D3 \end{vmatrix}$$

Where B1, C1, D1, ….are the respective cofactors of b1, c1, d1, … of determinant |A|.

For Example; let $|A| = \begin{vmatrix} b1 & c1 & d1 \\ b2 & c2 & d2 \\ b3 & c3 & d3 \end{vmatrix} = \begin{vmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{vmatrix}$

$$\text{Now, } B1 = (-1)^{1+1}\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = -2$$

$$B2 = (-1)^{2+1}\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = +1$$

$$B3 = (-1)^{3+1}\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = +4$$

Similarly, C1=4, C2= -2, C3=1 and D1=1, D2=4, D3= -2

Thus, $|A'| = \begin{vmatrix} -2 & 4 & 1 \\ 1 & -2 & 4 \\ 4 & 1 & -2 \end{vmatrix}$

Note: When $|A| \neq 0$, we have $|A'| = |A^2|$

Inverse Or Reciprocal Determinant

Suppose we have a determinant A, its inverse is represented as A^{-1}. Provided the $|A| \neq 0$, the inverse of A is formed by dividing every element of the adjoint of determinant A by |A|.

So, if $|A| = \begin{vmatrix} b1 & c1 & d1 \\ b2 & c2 & d2 \\ b3 & c3 & d3 \end{vmatrix}$, then $|A^{-1}| = \begin{vmatrix} B1/|A| & C1/|A| & D1/|A| \\ B2/|A| & C2/|A| & D2/|A| \\ B3/|A| & C3/|A| & D3/|A| \end{vmatrix}$

$$= \frac{1}{|A|.|A|.|A|} \begin{vmatrix} B1 & C1 & D1 \\ B2 & C2 & D2 \\ B3 & C3 & D3 \end{vmatrix}$$

$$= \frac{1}{|A|^3}. |A'| = \frac{1}{|A|^3}. |A^2| = |A|^{-1}$$

Symmetric Determinants

Suppose we consider the determinant $|A| = \begin{vmatrix} b11 & c12 & d13 \\ b21 & c22 & d23 \\ b31 & c32 & d33 \end{vmatrix}$, where the suffix values indicate the position of its respective element (i.e. b_{11} lies in the 1st row and 1st column and c_{12} lies in 1st row and 2nd column and so on). The general formula for a determinant is |A| $= \sum_{i=1}^{n} (\mathbf{a}rc)(\mathbf{f}rc)$ where $\mathbf{a}_{rc}$ are the elements of the determinant. A determinant is said to be symmetric if $\mathbf{a}_{rc} = \mathbf{a}_{cr}$ for all r,c = 1, 2, 3, …., n.

Thus, $\begin{vmatrix} b & c & d \\ c & e & f \\ d & f & a \end{vmatrix}$ is a symmetric determinant.

Properties of Symmetric Determinant:

1. If we find the adjoint of a symmetric determinant, we see that its adjoint is also symmetric.
2. If we square a symmetric determinant, the resultant determinant is also a symmetric determinant.

Skew and Skew-Symmetric Determinants

Suppose we consider the determinant $|A| = \begin{vmatrix} b11 & c12 & d13 \\ b21 & c22 & d23 \\ b31 & c32 & d33 \end{vmatrix}$, where the suffix values indicate the position of its respective element (i.e. b_{11} lies in the 1^{st} row and 1^{st} column and c_{12} lies in 1^{st} row and 2^{nd} column and so on). The general formula for a determinant is |A| $=\sum_{i=1}^{n}(\mathbf{a}rc)(\mathbf{f}rc)$ where $\mathbf{a}_{rc}$ are the elements of the determinant. A determinant is said to be 'skew' if $\mathbf{a}_{rc} = -\mathbf{a}_{cr}$ for all r,c = 1, 2, 3, …., n and r≠c.

Thus, $\begin{vmatrix} b & -c & -d \\ c & e & -f \\ d & f & a \end{vmatrix}$ is a skew determinant.

And if $\mathbf{a}_{rc} = -\mathbf{a}_{cr}$ for all r,c = 1, 2, 3, …., n and r≠c and $\mathbf{a}_{rc} = 0$ for all r=c, then the determinant is known as 'skew-symmetric'.

Thus, $\begin{vmatrix} 0 & -c & -d \\ c & 0 & -f \\ d & f & 0 \end{vmatrix}$ is a 'skew-symmetric' determinant.

Properties of Skew-Symmetric Determinant:

- Every determinant of 2^{nd} order (mostly even order) which is skew-symmetric is a perfect square.

  For Example, let a skew-symmetric determinant $|A| = \begin{vmatrix} 0 & 3 \\ -3 & 0 \end{vmatrix}$,

  Thus $|A| = 9 = 3^2$ which is a perfect square.

- Every determinant of 3^{rd} order (mostly odd order) which is skew-symmetric is zero.

  For Example, let a skew-symmetric determinant $|A| = \begin{vmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{vmatrix}$

  Thus, $|A| = 0 -(-1)(6) +(-2)(3) = 0$.

The Inverse of a Matrix

Suppose A be an non-singular matrix and the inverse matrix $B = A^{-1}$ exists such that $AB = BA = I$, where I is an identity matrix of order same as A or B.

A matrix ***A*** is non-singular iff $\det(A) = |A| \neq 0$. In case of singular matrix i.e. |A| = 0, the inverse does not exist.

For Example; $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \dfrac{d}{ad-bc} & -\dfrac{b}{ad-bc} \\ -\dfrac{c}{ad-bc} & \dfrac{a}{ad-bc} \end{pmatrix}$

The general formula for the inverse is given by;

Let A = (a_{ij}) be an n×n matrix with determinant $\det(A) = |A| \neq 0$ and it has a unique inverse A^{-1} such that A $A^{-1} = A^{-1}$ A=I, then;

$$A^{-1} = \frac{1}{|A|}.Adj(A)$$

Properties of the Inverse: Let A and B are invertible n×n matrix, then;

- If A^{-1} is invertible then $(A^{-1})^{-1} = A$
- If AB is invertible then $(AB)^{-1} = B^{-1}A^{-1}$
- If the transpose A' is invertible then $(A')^{-1} = (A^{-1})'$
- $(cA)^{-1} = c^{-1}A^{-1}$ whenever c is a number $\neq 0$

Finding Inverse by Elementary Row Operations

In this method, we can find the inverse of a matrix by row operation. It is known as elementary matrix method. It is given by;

$$[A \mid I] \text{ to } [I \mid A^{-1}]$$

It can be explained by the help of an example.

$$\text{Let } A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}, \text{ then}$$

$$[A|I] = \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$$

$$\approx \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right] \quad -2R_1 + R_2$$

$$\approx \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right] \quad -R_2$$

$$\approx \left[\begin{array}{cc|cc} 1 & 0 & -7 & 4 \\ 0 & 1 & 2 & -1 \end{array}\right] \quad -4R_2 + R_1$$

$$= [I|A^{-1}]$$

$$A^{-1} = \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}$$

Cramer's Rule and Simultaneous Equations

Using the properties of determinants, a simple method of solving linear simultaneous equations was proposed by the mathematician, Gabriel Cramer.

Suppose, we have following linear simultaneous equation:

$$b_1x + c_1y + d_1z = m_1$$

$$b_2x + c_2y + d_2z = m_2$$

$$b_3x + c_3y + d_3z = m_3$$

Now we can rewrite this equation form in the form of determinant as follows:

$$\begin{vmatrix} b1 & c1 & d1 \\ b2 & c2 & d2 \\ b3 & c3 & d3 \end{vmatrix} \times \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{vmatrix} m1 \\ m2 \\ m3 \end{vmatrix}$$

Now let $|A| = \begin{vmatrix} b1 & c1 & d1 \\ b2 & c2 & d2 \\ b3 & c3 & d3 \end{vmatrix} \neq 0$

Let $|B| = \begin{vmatrix} m1 & c1 & d1 \\ m2 & c2 & d2 \\ m3 & c3 & d3 \end{vmatrix}$

Now,

$$x \, . \, |A| = x. \begin{vmatrix} b1 & c1 & d1 \\ b2 & c2 & d2 \\ b3 & c3 & d3 \end{vmatrix} = \begin{vmatrix} xb1 & c1 & d1 \\ xb2 & c2 & d2 \\ xb3 & c3 & d3 \end{vmatrix}$$

$$= \begin{vmatrix} xb1 + yc1 + zd1 & c1 & d1 \\ xb2 + yc2 + zd2 & c2 & d2 \\ xb3 + yc3 + zd3 & c3 & d3 \end{vmatrix}$$

$$= \begin{vmatrix} m1 & c1 & d1 \\ m2 & c2 & d2 \\ m3 & c3 & d3 \end{vmatrix} = |B|$$

Therefore, $X \times |A| = |B|$

Thus, $x = \frac{|B|}{|A|}$.

Similarly we can solve for y and z values.

Let $|C| = \begin{vmatrix} b1 & m1 & d1 \\ b2 & m2 & d2 \\ b3 & m3 & d3 \end{vmatrix}$ and $|D| = \begin{vmatrix} b1 & c1 & m1 \\ b2 & c2 & m2 \\ b3 & c3 & m3 \end{vmatrix}$

Thus, from above we can say:

$$y = \frac{|C|}{|A|} \text{ and } z = \frac{|D|}{|A|}.$$

This is the process of solving simultaneous equations using Cramer's Rule.

For Example, Let $\begin{matrix} 3x+2y=-1 \\ 5x-3y=11 \end{matrix}$ be the Simultaneous Equations then

$$D = \begin{vmatrix} 3 & 2 \\ 5 & -3 \end{vmatrix} = (-9-10) = -19 \quad D_x = \begin{vmatrix} -1 & 2 \\ 11 & -3 \end{vmatrix} = (3-22) = -19 \quad D_y = \begin{vmatrix} 3 & -1 \\ 5 & 11 \end{vmatrix} = (33+5) = 38$$

Now, $x = \dfrac{D_x}{D} = \dfrac{-19}{-19} = 1$ and $y = \dfrac{D_y}{D} = \dfrac{38}{-19} = -2$

Example: Suppose there are two simultaneous equations:

a – 2b = 3 and 3a + 5b = 20

Find the values of a and b using Cramer's rule.

Solution: We can write the above equations in determinants form as follows:

AX = B where $A = \begin{vmatrix} 1 & -2 \\ 3 & 5 \end{vmatrix}$, $X = \begin{vmatrix} a \\ b \end{vmatrix}$ and $B = \begin{vmatrix} 3 \\ 20 \end{vmatrix}$

Let $A_1 = \begin{vmatrix} 3 & -2 \\ 20 & 5 \end{vmatrix}$ and $A_2 = \begin{vmatrix} 1 & 3 \\ 3 & 20 \end{vmatrix}$

According to Cramer's Rule,

$$a = \frac{|A1|}{|A|} = \frac{\begin{vmatrix} 3 & -2 \\ 20 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & 5 \end{vmatrix}} = \frac{(3*5)-(-2*20)}{(1*5)-(-2*3)} = 55/11 = 5$$

$$b = \frac{|A2|}{|A|} = \frac{\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & 5 \end{vmatrix}} = \frac{(1*20)-(3*3)}{(1*5)-(-2*3)} = 11/11 = 1$$

Homogeneous System of Equations

If the right hand side of the systems of equations consists only of zeros then this system is called homogeneous. This system will always have the so-called **trivial solution i.e.** $X_1 = X_2 = X_3$ = -------------- = Xn = 0. On the other side some homogeneous system has **nontrivial** solutions. The homogeneous linear system of equations has **nontrivial** solutions if and only if the coefficient matrix A = $(a_{ij})_{nxn}$ is singular, i.e., $|A| = 0$.

Solved Examples

Example: Show that $(AB)^{-1} = B^{-1}A^{-1}$

Solution: We can consider that

$$AB(AB)^{-1} = I$$
$$A^{-1}AB(AB)^{-1} = A^{-1}I = A^{-1}, \text{ multiplying by } A^{-1}$$
$$B^{-1}IB(AB)^{-1} = B^{-1}A^{-1}, \text{ using } A^{-1}A = I \text{ and multiplying by } B^{-1}$$
$$B^{-1}B(AB)^{-1} = B^{-1}A^{-1}$$
$$(AB)^{-1} = B^{-1}A^{-1}$$

Example: Consider the following macro-economic model;

$Y = C + I_0 + G_O$ and $C = a + bY$

Find Y (National Income) and C (Consumption) by using Cramer's rule.

Solution: Given;

$$Y - C = I_0 + G_O$$

$$-bY + C = a$$

Now, write the above equation in matrix form then

$$D = \begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix} = 1 - b = (1-b)$$

$$D_y = \begin{vmatrix} I_o + G_o & -1 \\ a & 1 \end{vmatrix} = (I_o + G_o) - (-1 \times a) = (I_o + G_o) + a$$

$$D_c = \begin{vmatrix} 1 & I_o + G_o \\ -b & a \end{vmatrix} = a + b(I_o + G_o)$$

then

$$Y = \frac{(I_o + G_o) + a}{(1-b)}$$

$$C = \frac{a + b(I_o + G_o)}{(1-b)}$$

Example: Prove that $|A| = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Solution: Given; $|A| = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

Applying $C_1 = C_1 - C_2$ and $C_2 = C_2 - C_3$, we get

$$|A| = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix} = (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a+b & b+c & c^2 \end{vmatrix}$$

Now, expanding the determinant by the first row, we get

$$|A| = (a-b)(b-c)\begin{vmatrix} 1 & 1 \\ a+b & b+c \end{vmatrix}$$

$$|A| = (a-b)(b-c)(b+c-a-b) = (a-b)(b-c)(c-a)$$

Example: Show that, A = [I - X(X'X)$^{-1}$X'] is idempotent.

Solution: Idempotent means, A^2 = A, then

A^2 = [I-X(X'X)$^{-1}$X']^2

= I^2 + X(X'X)$^{-1}$X'×X(X'X)$^{-1}$X'- I×X(X'X)$^{-1}$X' - X(X'X)$^{-1}$X'×I

= I +X(X'X)$^{-1}$X' - 2X(X'X)$^{-1}$X'

= [I - X(X'X)$^{-1}$X'] = A Proved

Example: A monomial square matrix M is one in which there is exactly one non-zero entry in each row and in each column. Show that any 2×2 monomial matrix is invertible and describe its inverse.

Solution: A monomial square matrix M must be one of two types, i.e

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} or \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \text{with } a \neq 0 \text{ and } b \neq 0$$

In both cases $|M| \neq 0$, so M is invertible.

And its inverse is given by;

$$M^{-1} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^{-1} and \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1/b \\ 1/a & 0 \end{pmatrix}$$

Example: For what value of μ the following system of equations has non-trivial solutions?

$$5x + 2y + z = \mu x$$
$$2x + y = \mu y$$
$$x + z = \mu z$$

Solution: Rewrite the given equation in standard form;

$$(5-\mu)x + 2y + z = 0$$
$$2x + (1-\mu)y = 0$$
$$x + (1-\mu)z = 0$$

The above system of equations has a nontrivial solution iff the coefficient matrix is singular i.e. the determinant of coefficient matrix must be zero.

$$\begin{vmatrix} 5-\mu & 2 & 1 \\ 2 & 1-\mu & 0 \\ 1 & 0 & 1-\mu \end{vmatrix} = 0$$

Now, expanding the determinant, we get the value of determinant

$$\mu(1-\mu)(\mu-6) = 0$$

Hence the system of equation has non-trivial solutions iff $\mu = 0$, 1 or 6

PROBLEM SET

1. Prove that $\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix} = (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)(\alpha-\beta)$

2. Solve the following system using both Cramer's rule and matrix inverse:

a) $\begin{cases} 2x-3y=3 \\ 4x-y=11 \end{cases}$

b) $\begin{cases} 5a-6b+4c=15 \\ 7a+4b-3c=19 \\ 2a+b+6c=46 \end{cases}$

d) $\begin{cases} 2x+3y-z+12=0 \\ 3x+4y+11z=46 \\ 5y+4z=5 \end{cases}$ e) $\begin{cases} 2x-y=5 \\ 3x+2y=-3 \end{cases}$

f)

X + 24 + 3z = 6

2x + 4y + z = 7

3x + 24 + 9z = 14

3. Calculate the determinant of $\begin{vmatrix} 2 & 0 & -1 \\ 1 & 1 & 7 \\ 3 & 3 & 9 \end{vmatrix}$,

4. Find the cofactors of the following determinant and prove that |A'| = $|A^2|$

$$|A| = \begin{vmatrix} 0 & 3 & 0 \\ 1 & 4 & 7 \\ 2 & 5 & 8 \end{vmatrix}.$$

5. For a given matrix $A = \begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}$, the transpose is $A' = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$. A matrix A is called orthogonal if $AA' = A'A = I$. Show that the matrix $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ is orthogonal.

6. Given an input coefficient matrix $A = \begin{bmatrix} \frac{240}{1200} & \frac{750}{1500} \\ \frac{720}{1200} & \frac{450}{1500} \end{bmatrix}$ and the demand matrix

$D = \begin{bmatrix} 210 \\ 330 \end{bmatrix}$. Find the output matrix *X*, such that $(I - A)X = D$

7. Find *k* so the system has no solution: $\begin{cases} 2x - 3y = 3 \\ kx - y = 11 \end{cases}$

8. Show that the following system of equations has no solution;

$$\begin{aligned} x + 2y + z &= 5 \\ 3x - y + z &= 2 \\ x - 5y - z &= 4 \end{aligned}$$

9. If A and B are the invertible then prove that $|BAB^{-1}| = |A|$

10. Find the inverse of the matrix $A = \begin{pmatrix} 3 & 5 \\ 7 & -11 \end{pmatrix}$ and verify that $A.A^{-1} = A^{-1}.A = I$

11. If $A = \begin{pmatrix} 3 & 5 \\ 2 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, then prove that $(AB)^{-1} = B^{-1}A^{-1}$

12. Prove that the homogenous system of equations

$$\begin{aligned} ax + by + cz &= 0 \\ bx + cy + az &= 0 \\ cx + ay + bz &= 0 \end{aligned}$$

Has a nontrivial solution if and only if $a^3 + b^3 + c^3 - 3abc = 0$

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